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JEL classification: C11, C13, G15

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Flexible and Robust Modelling of Volatility Comovements: A Comparison of Two Multifractal Models*

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Abstract

Long memory (long-term dependence) of volatility counts as one of the ubiquitous stylized facts of financial data. Inspired by the long memory property, multifractal processes have recently been introduced as a new tool for modeling financial time series. In this paper, we propose a parsimonious version of a bivariate multifractal model and estimate its parameters via both maximum likelihood and simulation based inference approaches. In order to explore its practical performance, we apply the model for computing value-at-risk and expected shortfall statistics for various portfolios and compare the results with those from an alternative bivariate multifractal model proposed by Calvet et al. (2006) and the bivariate CC-GARCH of Bollerslev (1990). As it turns out, the multifractal models provide much more reliable results than CC-GARCH, and our new model compares well with the one of Calvet et al. although it has an even smaller number of parameters.

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1 Introduction

The volatility dynamics of financial returns are an inexhaustible source of inspiration for the development of stochastic processes that match their stylized facts. After the seminal proposals of GARCH and stochastic volatility models with additive structure of their volatility dynamics models with multiplicative structure have been introduced recently under the heading of multifractal models. Multifractal models (henceforth, MF) conceive volatility as a hierarchical product of heterogeneous components with different lifetimes. The essential new feature of MF processes is their ability of generating different degrees of long-term dependence in various powers of returns - a feature pervasively found in empirical financial data, cf. Granger and Joyeux (1980), Lo (1991), Ding et al. (1993), Lux (1996), Labato and Savin (1998) and Zumbach (2004). While the development of the multifractal approach goes back to Benoit Mandelbrot's work on the distribution of turbulent dissipation in the 1970s, its adaptation in finance got started only in the late nineties, cf. Mandelbrot et al. (1997).

So far, the development of multifractal models has been confined mostly to univariate settings. However, for many important purposes in empirical research multivariate settings are preferable. In particular, it is now well accepted that financial volatility exhibits strong comovements over time across assets and markets. This is particularly important when considering asset allocation, value-at-risk statistics and portfolio hedging strategies. Secondly, since the information on the source of long memory in the volatility process is quite limited, multivariate settings may provide additional insights into the factors responsible for long-term dependence. Motivated by these considerations, we propose a new multivariate multifractal model as an alternative to the one previously studied by Calvet et al. (2006). Although we confine ourselves to the bivariate case in this paper, extensions to more than two simultaneous series are, in principle, straightforward.

The rest of this paper is organized as follows: Section 2 provides a short introduction to multifractal models in finance. Section 3 introduces our new parsimonious bivariate multifractal process and compares it to the previously proposed alternative bivariate MSM model by Calvet et al. (2006). Section 4 provides details on parameter estimation (maximum likelihood and simulated likelihood) for both models and assesses the small sample performance of these estimators for the new model via Monte Carlo simulations. Empirical applications for risk management purposes of our bivariate MF process are reported in Section 5. Section 6 provides some concluding remarks.

2 Review of Univariate Multifractal Models

Financial markets display some similarities to fluid turbulence. For example, both turbulence and financial fluctuations are characterized by intermittency at all scales. A cascade of energy flux is known to occur from the large scale of injection to the small scale of dissipation, cf. Mandelbrot (1974) and Harte

(2001). In statistical physics, such "cascades" are modeled by multiplicative operations on probability measures. Mandelbrot et al. (1997) first introduced the multifractal apparatus into finance, adapting the approach of Mandelbrot (1974) to an asset-pricing framework. This multifractal model of asset returns (MMAR) assumes that asset returns r_t follow a compound process, in which an incremental fractional Brownian motion is subordinate to the cumulative distribution function of a multifractal measure. Calvet and Fisher (2002) have developed methods of statistical inference for this approach.

However, the practical applicability of MMAR suffers from its combinatorial nature, i.e. the non-causal nature of the time transformation and from its non-stationarity due to the inherent restriction to a bounded interval. These limitations have been overcome by the development of an iterative version of the MF model, the Markov-switching multifractal model (MSM), cf. Calvet and Fisher (2001, 2004) and Lux (2008). Both the model of Calvet et al. (2006) and the new one to be detailed in the next section, are bivariate extensions of the univariate MSM. In the univariate setting, asset returns r_t are given by:

$$r_t = \sigma \left(\prod_{i=1}^{k} M_t^{(i)} \right)^{1/2} \cdot u_t, \tag{1}$$

with u_t drawn from a standard Normal distribution $N(0,1)$ and instantaneous volatility being determined by the product of k volatility components or multipliers $M_t^{(1)}$, $M_t^{(2)}$ ..., $M_t^{(k)}$, and a constant scale parameter σ. Each volatility component is renewed at time t with probability γ_i depending on its rank within the hierarchy of multipliers or remains unchanged with probability $1-\gamma_i$. Calvet and Fisher (2001, 2004) propose to specify transition probabilities as:

$$\gamma_i = 1 - (1 - \gamma_1)^{(b^{i-1})}, \tag{2}$$

with parameters $\gamma_1 \in (0,1)$ and $b \in (1, \infty)$. This specification guarantees convergence of the discrete-time multifractal process to a limiting continuous-time version with random renewals of the multipliers. Estimation of this model, then, involves the parameters γ_1 and b, the scale factor σ as well as the parameters characterizing the distribution of the volatility components $M_t^{(i)}$.

Using the iterative version of the multifractal model instead of its combinatorial predecessor and confining attention to unit time intervals, the resulting dynamics of eq. (1) can also be seen as a particular version of a stochastic volatility model. With this rather parsimonious approach, one preserves the hierarchical structure of MMAR while dispensing with its restriction to a bounded interval. While this model is asymptotically "well-behaved" (i.e. it shares all the convenient properties of Markov-switching processes) it is still capable of capturing some important properties of financial time series, namely, volatility clustering and the power-law behaviour of the autocovariance function of absolute moments:

$$Cov(|r_t|^q, |r_{t+\tau}|^q) \propto \tau^{2d(q)-1}. \tag{3}$$

Note, however, that the power-law behavior of the MSM model holds only approximately in a preasymptotic range. Rather than displaying asymptotic power-law behavior of autocovariance functions in the limit $t \to \infty$ or divergence of the spectral density at zero, the Markov-switching MF model is rather characterized by only 'apparent' long-memory with an approximately hyperbolic decline of the autocorrelation of absolute powers over a finite horizon and exponential decline thereafter. In particular, approximately hyperbolic decline as expressed in eq. (3) holds only over an interval $1 \ll \tau \ll b^k$ with b the parameter of the transition probabilities of eq. (2) and k the number of hierarchical levels. Eq. (3) also implies that different powers of absolute returns have different decay rates in their autocovariances, i.e. different degree of (pre-asymptotic) long-term dependence. One should note that it is this characteristic that distinguishes MF models from other long memory processes, such as FIGARCH and ARFIMA models, which belong to the category of unifractal models, i.e. they have the same decay rate d for all moments q.

Although the multifractal model is a rather new tool in financial economics, various approaches have already been explored to estimate its parameters. The parameters of the combinatorial MMAR have been estimated via an adaptation of the scaling estimator and Legendre transformation approach from statistical physics. However, this approach has been shown to yield unreliable results, cf. Lux (2004). A broad range of more rigorous estimation methods have been developed for iterative MF processes. Calvet and Fisher (2001, 2004) propose maximum likelihood (ML) whose applicability is, however, confined to the case of discretely distributed multipliers. Lux (2008) proposes a Generalized Method of Moments (GMM) approach, which can be applied not only to discrete but also to continuous distributions of the volatility components. Calvet, Fisher and Thompson (2006) have introduced a Markov-Chain Monte Carlo method to estimate the parameters of a bivariate extension of the MSM model.

3 Bivariate Multifractal Models

Multivariate settings provide relatively more information for portfolio and risk management, so that there is a natural tendency to generalize volatility models from the univariate case to multivariate specifications. Multivariate GARCH models are available in a number of different specifications (cf. Bauwens et al., 2006), and multivariate stochastic volatility models have gained increasing attention recently (e.g. Liesenfeld and Richard, 2003). A certain drawback of these approaches is that they are highly parameterized. In particular, in a general specification of a multivariate GARCH model, that allows for a full set of links between covariances, the number of parameters increases with the fourth power of the number of time series to be modelled. Of course, such an approach becomes unpractical beyond a small number of simultaneous time series. Various restrictions of this general framework have been proposed such as the conditional constant correlation model (CCC) of Bollerslev (1990) that only requires to estimate the parameters of N (number of time series) univariate

GARCH models.

The entirely different formalization of volatility correlations in the new type of multifractal models might provide an alternative avenue to parsimonious multivariate volatility modelling. Calvet et al. (2006) were first to propose a bivariate MF specification, which has also been used in Idier (2008). However, as with other processes, there are different ways to expand the baseline univariate models. We present here a second parsimonious alternative. Essentially, Calvet et al. (2006) assume that the same hierarchical volatility process with k individual components applies to both time series, but obeys a different specification (in the sense of different realizations of distributional parameters) for each one of them. Our approach, in contrast, assumes that there is only a limited number of joint volatility components together with additional ones whose realizations are independent across time series. The joint components, however, are assumed to also have identical parameters. The detailed presentation of both formalizations will emphasize the difference between these non-nested alternatives.

To set the stage, we consider two financial returns series $r_{n,t}$ (for $n = 1, 2$), and assume that instantaneous volatility is composed of heterogenous frequencies. We model the bivariate asset returns r_t as

$$r_t = \sigma. * [g(M_t)]^{1/2} . * u_t. \tag{4}$$

Here, r_t, σ, and u_t are all bivariate vectors: $r_t = \begin{bmatrix} r_{1,t} \\ r_{2,t} \end{bmatrix}$, $\sigma = \begin{bmatrix} \sigma_1 \\ \sigma_2 \end{bmatrix}$, $u_t = \begin{bmatrix} u_{1,t} \\ u_{2,t} \end{bmatrix}$, and $.*$ denotes element by element multiplication. σ is the vector of constant scale parameters (the unconditional standard deviation); u_t is a 2×1 vector whose elements follow a bivariate standard Normal distribution, with an unknown correlation parameter ρ, and $g(M_t)$ is the vector of the products of multifractal volatility components, i.e.

$$g(M_t) = \begin{bmatrix} g(M_{1,t}) \\ g(M_{2,t}) \end{bmatrix}, \tag{5}$$

with each $g(M_{q,t})$ defined as in the univariate case:

$$g(M_{q,t}) = \prod_{i=1}^{k} M_{n,t}^{(i)}, \tag{6}$$

as the product of volatility components for series n, $M_{n,t}^{(i)}$ denoting the volatility component at frequency i of series n:

$$M_t^{(i)} = \begin{bmatrix} M_{1,t}^{(i)} \\ M_{2,t}^{(i)} \end{bmatrix}. \tag{7}$$

Bivariate model specification by Calvet, Fisher and Thompson (CFT, 2006):

In the CFT model, the $M_t^{(i)}$ are drawn from a bivariate Binomial distribution $M = (M_1,\ \ M_2)'$, with M_1 taking values $m_1 \in (1,2)$ and $2-m_1$, and M_2 taking values $m_2 \in (1,2)$ and $2-m_2$.

While the framework by CFT allows for variation of the correlation (say, ρ_m) between components M_1 and M_2, they report that a correlation ρ_m equal to one is never rejected in their empirical applications. We, therefore, restrict this parameter to unity to economize on the number of parameters to be estimated.

In addition, whether or not certain volatility components (new arrivals) are updated for the individual MF processes is governed by the transition probabilities γ_i, which are specified as in the univariate version, cf. eq. (2). The correlation of arrivals between the two series is characterized by a parameter $\lambda \in [0,1]$, i.e., the probability of a new arrival at hierarchy level i for one time series given a new arrival in the other time series is $(1-\lambda)\gamma_i + \lambda$. New arrivals are independent if $\lambda = 0$ and simultaneous if $\lambda = 1$.

Alternative bivariate specification (Liu/Lux):

In the following section, we introduce a simple alternative to the specification of CFT (2006) with a more parsimonious setting. It assumes that two time series share a certain number of joint cascade levels while the remaining ones are drawn independently of each other, i.e., the above parameter λ is not the same for all hierarchical levels, but is equal to one for a range of low-frequency components and equal to zero for the remaining high-frequency entries. The economic intuition is that part of the correlation between assets is due to joint common factors such as the stage of the business cycle and long-term macroeconomic indicators, while the high-frequency components might have more idiosyncratic sources.

In our alternative model, we, therefore, assume for the column vector $g(M_t)$ that

$$g(M_{n,t}) = \prod_{i=1}^{j} M_{n,t}^{(i)} \cdot \prod_{l=j+1}^{k} M_{n,t}^{(l)}, \qquad \text{with } M_{1,t}^{(i)} = M_{2,t}^{(i)} \qquad \text{for } 1 < i \leq j \quad (8)$$

that means, both time series share a number of j joint cascades that govern the strength of their volatility correlation. Consequently, the larger j, the higher the correlation between them. After j joint multipliers, each series has additional independent multifractal components. In contrast to CFM, we assume that both $M_{1,t}^{(i)}$, $M_{2,t}^{(i)}$ are drawn from the same binomial distribution $M_{n,t}^{(i)} \sim \{m_0, 2-m_0\}$. As a consequence, there is only one parameter to be estimated for the distribution of multipliers, m_0.

Furthermore, to constrain the space of parameters further, a restriction for the specification of the transition probabilities is imposed for both models. Namely, we impose the simple specification $\gamma_i = 2^{-(k-i)}\quad, i = 1, \ldots, k$. Lux (2008) reports that the univariate MSM model possesses sufficient flexibility in the remaining parameters so that its empirical performance is relatively little hampered by fixing these parameters.

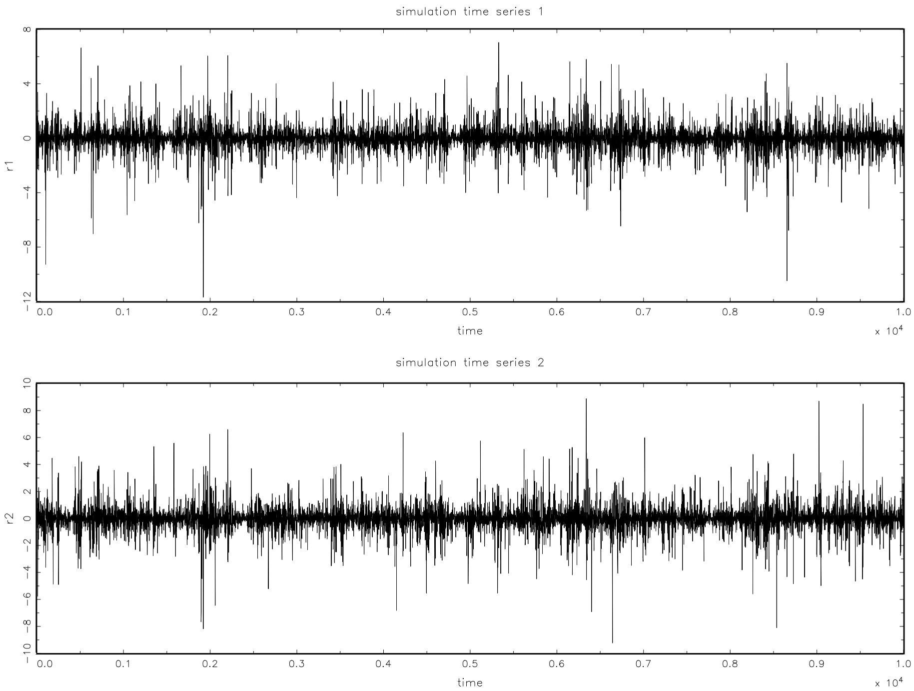

Figure 1: Simulation of the bivariate multifractal (Binomial) model. Specification and parameters are: $k = 20$, $j = 8$, $m_0 = 1.4$, $\rho = 0.5$, $\sigma_1 = \sigma_2 = 1$.

Figures 1 and 2 show simulations of this new bivariate multifractal model ($j = 4, k = 20$) with Binomial distribution of its multipliers together with the pertinent autocorrelation functions (ACF). The simulation apparently shares some of the stylized facts of financial time series, namely volatility clustering and hyperbolical decay of the autocorrelation function. One also easily recognizes the correlation of the volatility processes of both time series.

4 Parameter Estimation

Following Calvet and Fisher (2001,2004), Calvet, Fisher and Thompson (2006) and Lux (2008) we can adopt maximum likelihood, simulation-based likelihood and the generalized method of moments (GMM) for estimating the parameters of our new model. For further comparability with CFT, we restrict ourselves to ML and SML estimation in this paper, but we also note that the much less computationally demanding GMM approach has already been implemented as well in Liu (2008).

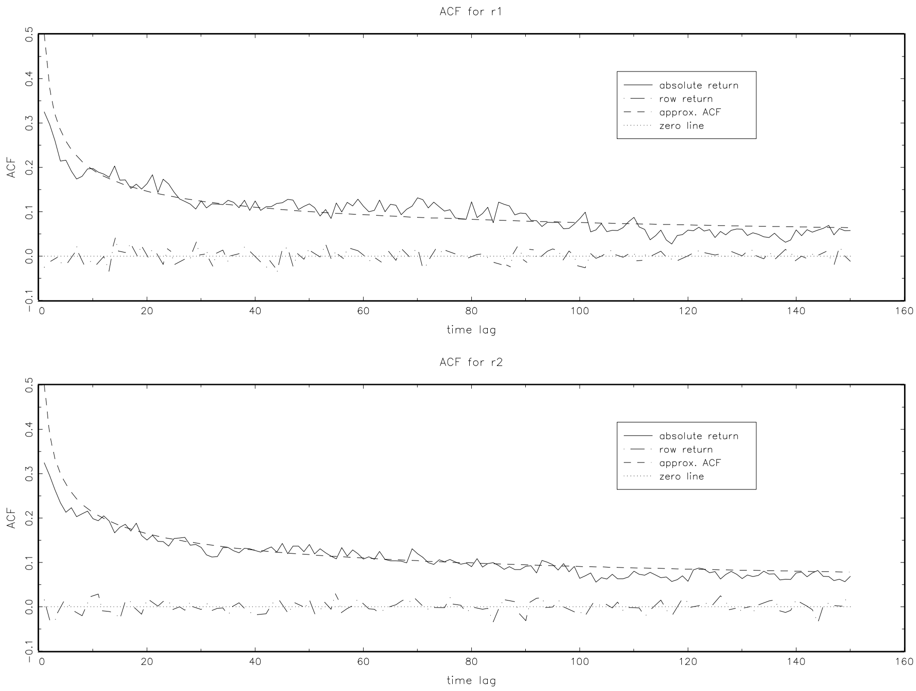

Figure 2: ACF for the simulations of the bivariate MF (Binomial) model of Figure 1.

4.1 Maximum Likelihood Estimation

Both multifractal models described above fall into the class of Markov-switching processes which makes maximum likelihood estimation feasible. As an advantage against more conventional Markov-switching models, MSM allows for very large state spaces without having to estimate large numbers of parameters. Since the state spaces are finite when the multipliers follow a discrete distribution (e.g. a Binomial distribution), the likelihood function can be obtained explicitly, cf. Calvet et. al (2006). For a sample size T, the likelihood function of both types of bivariate MSM models can, then, be written as:

$$\begin{aligned} L(r_1, \cdots, r_T; \theta) &= \prod_{t=1}^{T} f(r_t | r_1, \cdots, r_{t-1}) \qquad (9) \\ &= \prod_{t=1}^{T} \left[f(r_t | M_t = m^i) \cdot \sum_{i=1}^{4^k} P(M_t = m^i | r_1, \cdots, r_{t-1}) \right] \\ &= \prod_{t=1}^{T} f(r_t | M_t = m^i) \cdot (\pi_{t-1} A) \,. \end{aligned}$$

The likelihood function contains the following components: θ is the set of parameters to be estimated. The transition matrix A is composed of the conditional probabilities:

$$A_{ij} = \mathrm{P}(M_{t+1} = m^j | M_t = m^i). \tag{10}$$

On the right-hand side of eq. (10), M_t denotes the realization at time t of the bivariate Markov-switching process of the multipliers and m^i is an element of an ordered list of all possible realizations. Note that $i, j = 1, 2 \dots 4^k$ as there are 4^k different combinations of the realization of the multipliers of the bivariate model so that the transition matrix A has the dimension of $4^k \times 4^k$.[1]

The density of the innovation r_t conditional on M_t is:

$$f(r_t | M_t = m^i) = \frac{F_N \left\{ r_t./ \left[\sigma_q. * \eta^{1/2} \right] \right\}}{\sigma_q. * \eta^{1/2}}. \tag{11}$$

$F_N\{\cdot\}$ denotes the bivariate standard Normal density function, ./ represents element-by-element division, and $\eta = g(M_t)$. The last unknown element in the likelihood function is π_t, which is the vector of conditional probabilities, defined as $\pi_t^i = P(M_t = m^i | r_1, \cdots, r_t)$. By Bayesian updating, we get

$$\pi_t^i = \frac{f(r_t | M_t = m^i). * (\pi_{t-1} A)}{\sum_i f(r_t | M_t = m^i). * (\pi_{t-1} A)}. \tag{12}$$

In order to get an impression of the performance of the ML estimator for sample sizes typical for financial data, we have implemented the ML estimation for the Liu/Lux model with $k = 6$ and $j = 4$, and parameter values of $m_0 = 1.4$, $\rho = 0.5$, $\sigma_1 = 1$, $\sigma_2 = 1$. We conducted 400 simulations with sample sizes of $2,000$, $5,000$ and $10,000$. Table 1 reports key statistics for these simulations. As can be seen, the average bias of the Monte Carlo estimates is close to zero throughout different sample sizes. FSSE (finite sample standard error) and RMSE (root mean squared error) are quite small even for the small sample size $N = 2000$, and they are decreasing with increasing sample size. Simulations with different settings for $m_{0,\dots}, \rho, j$ and k (within the limits of fensibility for the overall number of components, k) are pretty similar and overall results are also quite similar to those reported for the CFT model.[2] Unfortunately, applicability of the ML approach is constrained by its computational demands: first, it is not applicable to models with an infinite state space, i.e. continuous distributions of the volatility components, such as the Lognormal distribution. Second, even for discrete distributions, current computational limitations make choices of cascades with a number of steps k beyond about 6 unfeasible because of the implied evaluation of a $4^k \times 4^k$ transition matrix in each iteration.

[1] Some of these have a probability of occurrence of zero in our new model because of the assumption of identical realizations for both series at levels $\leq k$. This feature also makes if computationally somewhat less burdensome than CFT.

[2] Liu (2008) also compares the performance of ML and SML estimations for both specifications and finds very similar small-sample performance.

Table 1: ML estimation for the Liu/Lux model

		N_1	N_2	N_3
Bias	$\hat{m}_0$	0.012	0.009	0.002
	$\hat{\sigma}_1$	0.016	0.012	0.006
	$\hat{\sigma}_2$	0.009	-0.013	0.005
	$\hat{\rho}$	0.011	-0.003	-0.007
FSSE	$\hat{m}_0$	0.03	0.017	0.006
	$\hat{\sigma}_1$	0.033	0.023	0.011
	$\hat{\sigma}_2$	0.035	0.021	0.012
	$\hat{\rho}$	0.026	0.017	0.007
RMSE	$\hat{m}_0$	0.032	0.017	0.007
	$\hat{\sigma}_1$	0.034	0.022	0.014
	$\hat{\sigma}_2$	0.035	0.022	0.014
	$\hat{\rho}$	0.029	0.018	0.008

Note: Simulations are based on the Liu/Lux model with the number of cascade levels $k = 6$ and $j = 4$, other parameters are $m_0 = 1.4$, $\rho = 0.5$, $\sigma_1 = 1$, $\sigma_2 = 1$. Sample lengths are $N_1 = 2,000$, $N_2 = 5,000$ and $N_3 = 10,000$. 400 Monte Carlo simulations have been carried out for each setting.

4.2 Simulation-Based Maximum Likelihood

Given the limitations of full maximum likelihood, it seems worthwhile to explore the performance of approximations to it that reduce the computational demands. One alternative proposed for CFT is a simulated ML approach using a particle filter algorithm. This seems particularly promising as the simulated particles can also be used to obtain a projection of the density into the future. We, therefore, proceed by adopting this SML approach for our alternative MSM model.

Here, we use a so-called particle filter, which is a class of simulation-based filters that recursively approximate the filtering of a random variable by a finite number of particles, i.e. discrete realizations sampled from the prior.

The conditional state probabilities can, then, be approximated by the average of B independently sampled particles:

$$\pi_t^i \propto f(r_t|M_t = m^i)\frac{1}{B}\sum_{b=1}^{B} P(M_t = m^i|M_{t-1} = m^{(b)}), \tag{13}$$

with $m^{(b)}$ the state of particle b, $b = 1, \ldots, B$. To generate the particles, we adopt the sampling/importance resampling (SIR) algorithm proposed by Pitt and Shephard (1999). This algorithm generates $\{M_t^{(b)}\}_{b=1}^{B}$ recursively. Denote by $M_t^{(b)}$ the bivariate realizations of the volatility components of particle no. b.

Given the population of particles of the previous period $\{M_{t-1}^{(b)}\}_{b=1}^{B}$, and new information on returns, r_t, one first generates B realisations $\hat{M}_t^{(1)}, \ldots \hat{M}_t^{(B)}$, drawing random numbers q from 1 to B with probabilities of

$$P(q = b) = \frac{f(r_t|\hat{M}_{t-1}^{(b)})}{\sum_{i=1}^{B} f(r_t|\hat{M}_{t-1}^{(i)})}. \tag{14}$$

Table 2: Simulation-based maximum likelihood estimation

SML										
		b=250			b=500			b=1000		
		N_1	N_2	N_3	N_1	N_2	N_3	N_1	N_2	N_3
Bias	$\hat{m}_0$	0.002	-0.013	-0.006	-0.012	-0.006	0.004	-0.005	0.005	0.003
	$\hat{\sigma}_1$	0.013	0.011	0.004	0.011	0.01	0.005	0.011	0.005	-0.003
	$\hat{\sigma}_2$	0.009	0.01	-0.006	0.01	-0.012	0.006	0.016	-0.01	-0.004
	$\hat{\rho}$	-0.015	-0.007	0.008	-0.007	0.007	-0.002	-0.006	-0.01	-0.001
FSSE	$\hat{m}_0$	0.033	0.017	0.008	0.03	0.018	0.007	0.03	0.017	0.006
	$\hat{\sigma}_1$	0.038	0.026	0.014	0.035	0.024	0.012	0.034	0.022	0.01
	$\hat{\sigma}_2$	0.04	0.028	0.013	0.038	0.026	0.012	0.035	0.026	0.012
	$\hat{\rho}$	0.028	0.016	0.008	0.025	0.016	0.007	0.026	0.016	0.007
RMSE	$\hat{m}_0$	0.034	0.019	0.009	0.032	0.017	0.009	0.031	0.016	0.007
	$\hat{\sigma}_1$	0.038	0.027	0.015	0.036	0.025	0.013	0.035	0.023	0.013
	$\hat{\sigma}_2$	0.041	0.027	0.014	0.038	0.025	0.014	0.036	0.025	0.012
	$\hat{\rho}$	0.031	0.019	0.009	0.027	0.018	0.007	0.028	0.017	0.008

Note: Simulations are based on the Liu/Lux model with the number of cascade levels $k = 6$ and $j = 4$, other parameters are $m_0 = 1.4$, $\rho = 0.5$, $\sigma_1 = 1$, $\sigma_2 = 1$. Sample lengths are $N_1 = 2,000$, $N_2 = 5,000$ and $N_3 = 10,000$. 400 Monte Carlo simulations have been carried out for each set of parameters.

Subsequently, one iterates these particles $\hat{M}_t^{(b)}$ applying the matrix of transition probabilities to obtain $M_t^{(b)} = \hat{M}_t^{(b)}$, $b = 1, \ldots, B$. This procedure replaces the extremely high dimensional state space evaluation of eq. (9) by a smaller number of Monte Carlo draws.

Table 3: Simulation-based maximum likelihood estimation

k=8, j=6										
		b=250			b=500			b=1000		
		N_1	N_2	N_3	N_1	N_2	N_3	N_1	N_2	N_3
Bias	$\hat{m}_0$	0.011	-0.010	-0.013	-0.007	-0.004	-0.009	-0.013	-0.009	-0.015
	$\hat{\sigma}_1$	-0.013	0.008	0.008	-0.012	-0.008	0.010	-0.008	0.013	-0.008
	$\hat{\sigma}_2$	0.009	0.013	-0.007	0.021	0.015	-0.008	0.008	-0.007	-0.010
	$\hat{\rho}$	-0.014	-0.008	0.010	-0.008	-0.009	0.010	-0.012	0.010	-0.011
SD	$\hat{m}_0$	0.047	0.023	0.012	0.038	0.020	0.011	0.035	0.020	0.008
	$\hat{\sigma}_1$	0.053	0.039	0.019	0.043	0.034	0.017	0.041	0.030	0.015
	$\hat{\sigma}_2$	0.054	0.037	0.019	0.046	0.032	0.018	0.045	0.031	0.016
	$\hat{\rho}$	0.035	0.020	0.014	0.030	0.018	0.010	0.030	0.017	0.009
RMSE	$\hat{m}_0$	0.048	0.025	0.012	0.039	0.020	0.012	0.035	0.021	0.009
	$\hat{\sigma}_1$	0.054	0.040	0.019	0.045	0.034	0.018	0.042	0.031	0.016
	$\hat{\sigma}_2$	0.054	0.038	0.021	0.046	0.032	0.019	0.044	0.030	0.016
	$\hat{\rho}$	0.037	0.021	0.013	0.031	0.018	0.009	0.031	0.017	0.010
k=8, j=4										
		b=250			b=500			b=1000		
		N_1	N_2	N_3	N_1	N_2	N_3	N_1	N_2	N_3
Bias	$\hat{m}_0$	-0.012	-0.016	-0.007	0.009	-0.007	-0.005	0.012	-0.010	-0.011
	$\hat{\sigma}_1$	0.008	0.013	0.006	-0.011	0.007	0.012	0.009	-0.005	-0.007
	$\hat{\sigma}_2$	-0.011	-0.003	0.011	-0.012	-0.005	0.005	-0.007	-0.011	-0.004
	$\hat{\rho}$	-0.007	-0.008	-0.010	-0.005	-0.008	-0.009	-0.018	-0.010	-0.010
SD	$\hat{m}_0$	0.048	0.024	0.011	0.037	0.021	0.012	0.035	0.020	0.009
	$\hat{\sigma}_1$	0.055	0.039	0.020	0.043	0.033	0.018	0.042	0.030	0.016
	$\hat{\sigma}_2$	0.054	0.038	0.020	0.045	0.033	0.018	0.044	0.031	0.015
	$\hat{\rho}$	0.036	0.021	0.014	0.032	0.018	0.009	0.030	0.017	0.010
RMSE	$\hat{m}_0$	0.050	0.028	0.014	0.040	0.021	0.011	0.036	0.021	0.010
	$\hat{\sigma}_1$	0.054	0.041	0.020	0.044	0.034	0.017	0.043	0.031	0.016
	$\hat{\sigma}_2$	0.055	0.038	0.023	0.045	0.032	0.019	0.043	0.030	0.016
	$\hat{\rho}$	0.038	0.021	0.012	0.032	0.018	0.009	0.030	0.016	0.011

Note: Simulations are based on the bivariate MF model with the number of cascade levels $k = 8$ and $j = 6(4)$, other parameters are $m_0 = 1.4$, $\rho = 0.5$, $\sigma_1 = 1$, $\sigma_2 = 1$. Sample lengths are $N_1 = 2,000$, $N_2 = 5,000$ and $N_3 = 10,000$. 400 Monte Carlo simulations have been carried out for each set of parameters.

We have also implemented the above SML algorithm in simulations along the lines of the previous ML evaluation. We first implemented the simulation-based ML with the same parameter settings used for the previous ML estimation exercise and the same time series so that all differences from the results in Table 1 are due to the different estimation methods. Again, 400 simulations and estimations were carried out with 100,000 observation generated in each simulation, of which, three different sizes of sub-samples ($N_1 = 2000$, $N_2 = 5000$, and $N_3 = 10000$) were randomly selected for estimation, and different numbers of particles $B = 250$, $B = 500$, $B = 1000$ were employed to implement the SIR algorithm. The pertinent Monte Carlo results are reported in Table 2. In addition, SML has also been tested with a larger number of cascade levels of $k = 8$ to explore the performance in cases that are not practically accessible for full ML estimation. Table 3 provides results for different numbers of joint cascade levels: $j = 6$, and $j = 4$, respectively. We observe that the bias is minor throughout different sub-sample sizes. FSSE (finite-sample standard error) and RMSE (root mean squared error) are relatively moderate and are decreasing with increasing sub-sample sizes. The major insights to be drawn from these experiments are the following: First comparing Tables 1 and 2, we see that even for only 250 particles, the loss in efficiency from SML against full ML is very small. Increasing the number of particles to 1000, there is practically no noticable difference in the performance of both algorithms any more. This nice behavior of SML is confirmed by the results exhibited in Table 3. Although one might intuitively expect a certain sensitivity of the particle filter to the number of possible states, there is hardly any noticable deterioration for higher k. Although the number of states is larger than the number of particles, the known potential inefficiencies of the particle filter (reduction of effective size of the swarm or 'sample impoverishment' due to the large weights of few particles) appear of little concern for our application. Further experiments with a large number of alternative settings confirm that Tables 2 and 3 are typical for the performance of the SML algorithm in our framework.

Note, however, that our Monte Carlo experiments have been conducted under the assumption of known numbers of joint and overall volatility components, j and k. Of course, in practice, these numbers are not known but also have to be estimated. Since different j and k imply comparison of non-nested models with very similar observational features, the problem of optimal determination of those parameters is a relatively hard task. Calvet and Fisher (2004) develop a test of the statistical significance of likelihood differences, but only find significant results for alternatives that are far apart from each other (in terms of k). Lux (2008), however, indicates that misspecification of k beyond some relatively high numbers might provide little practical harm to parameter estimation and applications in volatility forecasting and risk management. The reason is that for increasingly higher k, the additional components would have a relatively long mean life time (compared to the sample size) and would, therefore, be almost constant for the sample size under consideration. Indeed, parameter estimates typically show saturation and do not change anymore when increasing k of univariate MSM models beyond about 10. Both Lux (2008) and Lux and Morales-Arias (2010) report that for most practical purposes, the MSM model is insensitive towards misspecification with too high numbers of cascades.

In our case, the task of model selection is complicated by the fact that we have to make a choice of both j (number of joint cascades) and k (overall number of cascades. For the sake of computational feasibility, we fix the latter at $k = 8$. For selection of j we adopt a heuristic approach that will be detailed in the next section.

4.3 Model Selection

As has been mentioned, discrimination between different MF specifications on the base of explicit tests (like the likelihood ratio test applied in Calvet and Fisher, 2004) often remains inconclusive. Unlike for many other classes of models, there is also no general principle of parsimonious model selection that could be adopted for MSM models. Whereas, for example, a lower number of lags would lead to a large saving in terms of parameters to be estimated for GARCH or VAR models, this is not so for the number of multipliers in our MSM framework. Indeed, the number of parameters one has to estimate remains exactly the same for all choices of j and k. Hence, the standard information criteria are not applicable to our case. In light of these problems, we have turned to a heuristic model selection algorithm, that focuses on the proximity of the chosen model to important empirical characteristics of financial data. In particular, we believe that it should be crucial for practical applications to get a close approximation to the pattern of dependence of the data. Our proposed approach to subset model selection, therefore, focuses on this aspect. While $k = 8$ is predetermined for reasons of computability, the submodel for the number of joint cascades j is chosen depending on a comparison of the decay factor of the autocorrelation function of a portfolio of two assets with the pertinent statistics obtained from simulations. Our algorithm proceeds in the following way:

(1) we estimate our bivariate models for a range of joint cascade levels $j = 1, \ldots, 7$. Based on the empirical estimates of the different submodels, 200 simulations are conducted for each asset,

(2) for each bivariate time series, we take its equal-weighted portfolio and using the GPH approach of Geweke and Porter-Hudak (1983), estimate the empirical long memory decay parameter $\hat{d}$ for the absolute value of returns of the portfolio,

(4) we also compute the long memory parameter $\hat{d}_i$, $i = 1, \ldots, 200$ for each simulated equal-weighted portfolio. We then select the number of joint cascades, j, as the one for which the mean value of the 200 $\hat{d}_i$'s, is closest to the empirical one.

Table 4: Empirical estimates (in sample data)

	$j=1$	$j=2$	$j=3$	$j=4$	$j=5$	$j=6$	$j=7$
Dow/Nik							
$\hat{m_0}$	1.365	1.335	1.392	1.371	1.331	1.324	1.315
	(0.018)	(0.019)	(0.017)	(0.014)	(0.013)	(0.017)	(0.019)
$\hat{\sigma_1}$	0.974	0.952	1.032	0.947	0.954	0.937	1.021
	(0.019)	(0.019)	(0.021)	(0.018)	(0.025)	(0.023)	(0.025)
$\hat{\sigma_2}$	0.994	1.093	1.071	1.093	1.094	0.998	1.047
	(0.026)	(0.020)	(0.024)	(0.029)	(0.030)	(0.031)	(0.030)
$\hat{\rho}$	0.162	0.141	0.153	0.160	0.156	0.150	0.143
	(0.017)	(0.014)	(0.014)	(0.016)	(0.018)	(0.018)	(0.015)
$\hat{d}$	0.230	0.244	0.252	0.257	0.253	0.235	0.239
	(0.019)	(0.022)	(0.020)	(0.022)	(0.021)	(0.019)	(0.022)
US/DM							
$\hat{m_0}$	1.440	1.459	1.473	1.521	1.505	1.487	1.499
	(0.022)	(0.022)	(0.024)	(0.029)	(0.027)	(0.022)	(0.022)
$\hat{\sigma_1}$	0.790	0.745	0.778	0.780	0.744	0.739	0.785
	(0.019)	(0.022)	(0.019)	(0.020)	(0.015)	(0.021)	(0.021)
$\hat{\sigma_2}$	0.694	0.713	0.690	0.693	0.698	0.708	0.707
	(0.019)	(0.019)	(0.017)	(0.022)	(0.021)	(0.017)	(0.022)
$\hat{\rho}$	0.322	0.323	0.302	0.314	0.309	0.310	0.291
	(0.015)	(0.017)	(0.012)	(0.018)	(0.018)	(0.016)	(0.017)
$\hat{d}$	0.168	0.163	0.179	0.188	0.183	0.194	0.216
	(0.020)	(0.018)	(0.021)	(0.020)	(0.019)	(0.021)	(0.021)
TB2/TB1							
$\hat{m_0}$	1.561	1.633	1.610	1.602	1.592	1.610	1.685
	(0.024)	(0.021)	(0.025)	(0.025)	(0.028)	(0.027)	(0.031)
$\hat{\sigma_1}$	0.310	0.305	0.306	0.290	0.282	0.316	0.323
	(0.019)	(0.021)	(0.021)	(0.012)	(0.013)	(0.017)	(0.020)
$\hat{\sigma_2}$	0.345	0.358	0.363	0.356	0.307	0.380	0.369
	(0.031)	(0.027)	(0.027)	(0.028)	(0.028)	(0.029)	(0.021)
$\hat{\rho}$	0.880	0.869	0.858	0.844	0.812	0.845	0.858
	(0.029)	(0.023)	(0.023)	(0.022)	(0.026)	(0.026)	(0.026)
$\hat{d}$	0.155	0.141	0.145	0.137	0.126	0.139	0.138
	(0.017)	(0.017)	(0.015)	(0.018)	(0.016)	(0.020)	(0.019)

Note: Empirical estimates are obtained via a particle filter with $B = 500$. Each column corresponds to estimates with different joint numbers of cascade levels j ($k = 8$); $\hat{d}$ is the mean value of 200 simulated GPH estimates on the base of Monte Carlo simulations of the model with pertinent estimated parameters, and numbers in parenthesis are standard errors. The empirical GPH estimator $\hat{d}$ of *Dow/Nik* is 0.245; the empirical GPH $\hat{d}$ of *US/DM* is 0.177; the empirical GPH $\hat{d}$ of *TB2/TB1* is 0.129.

5 Empirical Applications

5.1 Parameter Estimation and Model Selection

In this section, we report our empirical results, and compare the performances of our new parsimonious bivariate MSM model with that of CFM and with the CC-GARCH model of Bollerslev (1990) as benchmarks. We consider daily data for two stock exchange indices: the Dow Jones composite 65 average index and the $NIKKEI$ 225 average index (DOW/NIK, 6^{th} January 1970 - 30^{th} December 2008), two foreign exchange rates, the U.S. Dollar to British Pound, and German Mark to British Pound (US/DM, 1^{st} March 1973 - 31^{st} December 2008); and a bond portfolio of U.S. 1-year and 2-year treasury constant maturity bond rates ($TB1/TB2$, 1^{st} June 1976 - 31^{st} December 2008), where the first symbol inside the parentheses gives the acronym for the corresponding time series, followed by the starting and ending dates for the sample at hand. Asset return are calculated as the log differences of prices $r_t = 100 \times (\log(p_t) - \log(p_{t-1}))$, with p_t denoting daily price observations.[3]

We separate each time series into two subsets (in-sample data used for estimation, out-of-sample data for forecast assessment). For the in-sample periods we use for DOW/NIK: 6^{th} January 1970 - 31^{st} August 1990; US/DM: 1^{st} March 1973 - 30^{th} April 1992; and $TB1/TB2$: 1^{st} June 1976 - 31^{st} May 1994. The remaining out-of-sample subsets are for the DOW/NIK: 4^{th} September 1990 - 30^{th} December 2008; US/DM: 1^{st} May 1992 - 31^{st} December 2008 and $TB1/TB2$: 1^{st} June 1994 - 31^{st} December 2008.

Table 4 provides the empirical estimates for different choices of joint cascade levels j ranging from 1 to 7 ($k = 8$), as well as the mean long memory GPH parameter for simulated portfolios (bottom). For the DOW/NIK portfolio the preferred model according to the heuristic model selection scheme detailed in sec. 4.3 is $j = 2$, while it is $j = 3$ for US/DM and $j = 5$ for $TB1/TB2$ respectively.

We have also conducted Monte Carlo experiments to further investigate the validity of the GPH selection procedure. After steps 1 to 3 above, that is, simulating 200 time series under the estimated parameters reported in Table 4 for each number of joint cascades, j, we obtain their GPH long memory indices $d_{i,j}$ ($i = 1, 2, \cdots, 200$; $j = 1, 2, \cdots, 7$) as well as mean values $\bar{d}_j$ over the 200 replications for specification j (reported in the bottom of each panel in Table 4). We then compare each $d_{i,j}$ with the $\bar{d}_j$ and count the number of cases in which the $d_{i,j}$ are closest to each one of the $\bar{d}_j$. Results for the stock markets are depicted in Table 5. Results for bonds and foreign exchange rates are very similar. Typically, in about 50 percent of cases the correct model can be identified in this way. If we add the next higher and lower j we arrive at about 150 of 200 cases. Given the very small fluctuations of likelihood values, we view this as a relatively satisfactory performance of our heuristic approach.

[3] The U.S. one and two-year treasury constant maturity rates have been converted to equivalent bond prices before calculating returns.

Table 5: GPH specification

	$j'=1$	$j'=2$	$j'=3$	$j'=4$	$j'=5$	$j'=6$	$j'=7$
Dow/Nik							
$j=1$	123	27	12	9	11	16	2
$j=2$	22	127	10	24	16	0	1
$j=3$	20	5	106	10	28	21	10
$j=4$	25	22	13	115	10	12	3
$j=5$	17	6	11	20	111	22	13
$j=6$	23	5	16	10	27	112	7
$j=7$	23	10	16	11	43	8	89

Note: Each row corresponds to the number of cases in which the individual d_{ij} ($i = 1, 2, \cdots, 200; j = 1, 2, \cdots, 7$) are closest to each one of the $\bar{d}_{j'}$ (mean of 200 d for different joint cascades $j' = 1, 2 \cdots, 7$).

5.2 Application to Value-at-Risk

Value-at-risk (VaR) has emerged as one of the most prominent tools for the assessment of downside market risk. For instance, according to the Basle Committee (1996), the risk capital of a bank must be sufficient to cover losses on the banks' trading portfolio over a 10-day holding period on 99% of occasions. VaR, therefore, represents a quantile of an estimated profit-loss distribution. Existing methodologies for calculating VaR differ in a number of respects. The most widespread are non-parametric historical simulation methods which estimate VaR by using the sample quantile estimate based on historic return data, fully parametric methods based on econometric models for volatility dynamics which often impose certain distributional assumptions, and semi-parametric methods based on extreme value theory (EVT) focusing only on the tails of the return distribution. For surveys of the advantages and disadvantages of various VaR approaches, see Dowd (2002) and Kuester et al. (2006).

Here, we will of cource, explore the performance of our new bivariate MSM model in VaR applications. To fix notation, let I_t be the information set until time t, and $r_{t,t+h}$ the forward looking h-period return at time t. Value-at-risk at time t for the h-period horizon is defined as:

$$Pr\left(r_{t:t+h} \leq -VaR^{\alpha}_{t:t+h}|I_t\right) = \alpha. \quad (15)$$

It places an upper bound on losses in the sense that these should exceed the VaR threshold with only a pre-assumed target probability. In other words, conditional on the information up to time t, the value-at-risk for period h of one unit of the portfolio is the $(1-\alpha)$th quantile of the conditional distribution $r_{t:t+h}$.

The algorithm of the particle filter in Section 3 also provides us with a way of calculating $VaR^{\alpha}_{t:t+h}$. To this end, we simulate each volatility particle $M_t^{(b)}$ one-step-ahead by using SIR: After having estimated the parameters with in-sample data, we invoke once more the particle filter algorithm and iterate the particle swarm under given estimated parameters up to the end of the in-sample record. Since this requires less computation time than the estimation stage, we use a larger number of particles, $B = 10,000$, in order to improve the approximation to the projected bivariate density. After the last iteration of the in-sample series (time t), we simulate the Markov chain h-steps-ahead to obtain $\{\hat{M}^{(b)}_{t+1}\}^B_{b=1}, \ldots, \{\hat{M}^{(b)}_{t+h}\}^B_{b=1}$, which are used to approximate the distribution of one- to h-step ahead forecasts of volatility. With the swarm of particles having been iterated into the future $(t+1, \ldots, t+h)$ we generate additional bivariate Normal innovations to construct the Monte Carlo realizations of our portfolio returns. When moving the forecast origin from t to $t+1$, we again apply the SIR in order to extract the new information at $t+1$, and then iterate the Markov chain to generate draws $\{\hat{M}^{(b)}_{t+2}\}^B_{b=1}$ to $\{\hat{M}^{(b)}_{t+h+1}\}^B_{b=1}$ conditional on $\{M^{(b)}_{t+1}\}^B_{b=1}$.

This recursive procedure provides a discrete approximation to Bayesian updating of state probabilities, which are then used as the basis to simulate the bivariate series forward over h-day horizons. Given the approximation to the predictive bivariate density by our particles we calculate $VaR^{\alpha}_{t:t+h}$ as the $(1-\alpha)$th simulated quantile.

Table 6: SML estimates for Calvet/Fisher/Thompson model (in-sample data)

	$\hat{m}_1$	$\hat{m}_2$	$\hat{\sigma}_1$	$\hat{\sigma}_2$	$\hat{\rho}$	$\hat{\lambda}$
Dow/Nik	1.435	1.375	0.924	1.211	0.288	0.373
	(0.022)	(0.030)	(0.034)	(0.037)	(0.024)	(0.020)
US/DM	1.430	1.415	0.797	0.672	0.276	0.470
	(0.025)	(0.022)	(0.036)	(0.037)	(0.015)	(0.029)
TB1/TB2	1.371	1.447	0.357	0.411	0.804	0.519
	(0.029)	(0.032)	(0.043)	(0.042)	(0.020)	(0.024)

Note: The number of cascade levels is $k = 8$ as in Calvet et al (2006).

Table 7: CC-GARCH(1, 1) model estimates (in-sample data)

	$\hat{\omega}_1$	$\hat{\omega}_2$	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\rho}_{12}$
Dow/Nik	0.01	0.05	0.04	0.22	0.94	0.73	0.07
	(0.00)	(0.02)	(0.01)	(0.02)	(0.01)	(0.03)	(0.02)
US/DM	0.12	0.14	0.11	0.13	0.87	0.82	0.22
	(0.01)	(0.02)	(0.01)	(0.03)	(0.01)	(0.02)	(0.03)
TB1/TB2 -0.02	-0.01	0.07	0.09	0.85	0.83	0.71	
	(0.01)	(0.00)	(0.03)	(0.03)	(0.01)	(0.02)	(0.03)

Note: The ML estimation of the CC-GARCH(1, 1) model is implemented via the GAUSS module 'Fanpac' provided by Aptech ™ Systems Inc.

We also compute value-at-risk based on the alternative CFM multifractal model and on the bivariate constant correlation GARCH (CC-GARCH) model, of Bollerslev (1990), for comparison. Tables 6 and 7 report the empirical estimates of the parameters of the CFT and CC-GARCH models for the in-sample series, while the pertinent parameters of the Liu/Lux model can be found in Table 4. While value-at-risk for the CFM model is also computed via an SIR approach, for the CC-GARCH (1, 1) model, a closed form solution exists for one-day VaR forecasts, that is,

$$VaR^{\alpha}_{t:t+1} = \mu_t + Q_{1-\alpha}\sigma_t, \tag{16}$$

with $Q_{1-\alpha}$ the $(1-\alpha)$th quantile of the standard Normal distribution and σ_t is the square root of the conditional volatility (standard deviation) implied from CC-GARCH. VaR forecasts for more than one day are implemented again through simulations.

We assess the model's performances by computing the failure rate for the individual assets as well as equal weight portfolios and hedge portfolios over the out of sample VaR forecasts. By definition, the failure rate is the number of times losses exceed the forecasted VaR in a given sample, which should be close to the prescribed significance level if the model is correctly specified. We then perform Kupiec's likelihood ratio (LR) test, cf. Kupiec (1995). Since the computation of the empirical failure rate is characterized as a sequence of yes/no observations, the null hypothesis that the model generates the correct failure rate amounts to testing:

H_0: $\alpha = \hat{\alpha}$, against
H_1: $\alpha \neq \hat{\alpha}$,

where $\hat{\alpha}$ is the empirical failure rate estimated. The results under the alternative models are reported in Table 8 (Liu/Lux model), Table 9 (Calvet/Fisher/Thompson model) and Table 10 (CC-GARCH model) for horizons of one, two and five days. These results demonstrate that VaR forecasts based on multifractal processes are much more accurate those those based on CC-GARCH model. For example, out of a total of 36 scenarios for stock markets, there are only 4 too risky cases in the Liu/Lux model over one and five-day horizons, and 4 failures (three too risky and one too conservative) in the Calvet/Fisher/Thompson model for two and five-day horizons. In contrast, we find 9 too risky cases based on the CC-GARCH model. Forecasts of foreign exchange rates show even more encouraging results: we observe only one conservative scenario for the two-day horizon for the Liu/Lux model in Table 8, and two cases for the Calvet/Fisher/Thompson model in Table 9. However, there is a total of 13 too risky cases in Table 10 based on the CC-GARCH model, where most failures occur at the 1% level. For the U.S bond market, there are 7 failures and 8 unsuccessful forecasts in Table 8 and Table 9, respectively, whereas there are 14 failures for the CC-GARCH model.

A glance at the comparison suggests that the performances of multifractal models clearly dominates that of the CC-GARCH model, and it also shows that our parsimonious multivariate MF model with smaller number of parameters than CFM is able to achieve a similar performance of its VaR forecast.

Table 8: Failure rates for multi-period Value-at-Risk forecasts (Liu/Lux model)

		One day horizon				Two days horizon				Five days horizon			
		NIK	*DOW*	*EW*	*HG*	*NIK*	*DOW*	*EW*	*HG*	*NIK*	*DOW*	*EW*	*HG*
Stocks	$\alpha = 10\%$	0.1060	0.1191^+	0.1096	0.0954	0.0929	0.1106	0.1072	0.0986	0.0912	0.1214^+	0.1079	0.0932
	$\alpha = 5\%$	0.0428	0.0592	0.0554	0.0495	0.0476	0.0594	0.0556	0.0454	0.0438	0.0562	0.0534	0.0468
	$\alpha = 1\%$	0.0069	0.0175^+	0.0074	0.0184^+	0.0077	0.0084	0.0142	0.0053	0.0062	0.0128	0.0167	0.0062
		US	*DM*	*EW*	*HG*	*US*	*DM*	*EW*	*HG*	*US*	*DM*	*EW*	*HG*
FXs	$\alpha = 10\%$	0.0984	0.1005	0.0905	0.1161	0.1019	0.0972	0.1020	0.0960	0.1026	0.0961	0.0982	0.1077
	$\alpha = 5\%$	0.0501	0.0477	0.0415	0.0604	0.0529	0.0437	0.0453	0.0404	0.0508	0.0452	0.0515	0.0560
	$\alpha = 1\%$	0.0063	0.0090	0.0048	0.0141	0.0133	0.0062	0.0108	0.0025*	0.0039	0.0071	0.0127	0.0048
		TB1	*TB2*	*EW*	*HG*	*TB1*	*TB2*	*EW*	*HG*	*TB1*	*TB2*	*EW*	*HG*
Bonds	$\alpha = 10\%$	0.0898	0.1117	0.0970	0.0715*	0.0989	0.0959	0.1003	0.0740*	0.0993	0.1077	0.1049	0.0974
	$\alpha = 5\%$	0.0602	0.0513	0.0500	0.0481	0.0565	0.0587	0.0604	0.0352*	0.0436	0.0587	0.0517	0.0373*
	$\alpha = 1\%$	0.0116	0.0083	0.0080	0.0031*	0.0102	0.0078	0.0127	0.0049	0.0044	0.0125	0.0189^+	0.0019*

Note: This table shows the failure rate (proportion of observations above the VaR). Stocks are Dow Jones composite 65 average index (DOW) and NIKKEI 225 stock average index (NIK); FXs are foreign exchange rate of U.S. Dollar (US) and German Mark (DM) to British Pound; Bonds are the U.S. 1-year and 2-year treasury constant maturity rate (TB1, TB2 respectively). EW denotes equal-weight portfolio, HG denotes hedge, a zero investment portfolio. + and $*$ denote too risky and too conservative VaR, respectively. The standard errors are calculated according to the approach of Kupiec (1995).

Table 9: Failure rates for multi-period Value-at-Risk forecasts (Calvet/Fisher/Thompson model)

		One day horizon				Two days horizon				Five days horizon			
		NIK	*DOW*	*EW*	*HG*	*NIK*	*DOW*	*EW*	*HG*	*NIK*	*DOW*	*EW*	*HG*
Stocks	$\alpha = 10\%$	0.1046	0.0910	0.1107	0.0934	0.0901	0.1105	0.1063	0.1076	0.0988	0.1097	0.0920	0.0896
	$\alpha = 5\%$	0.0601	0.0538	0.0555	0.0441	0.0387*	0.0590	0.0607	0.0543	0.0497	0.0446	0.0503	0.0417
	$\alpha = 1\%$	0.0104	0.0068	0.0087	0.0044	0.0027*	0.0126	0.0076	0.0058	0.0024*	0.0171^{+}	0.0058	0.0056
		US	*DM*	*EW*	*HG*	*US*	*DM*	*EW*	*HG*	*US*	*DM*	*EW*	*HG*
FXs	$\alpha = 10\%$	0.0904	0.0937	0.1041	0.1102	0.1006	0.1033	0.0976	0.0983	0.1011	0.1005	0.0992	0.1079
	$\alpha = 5\%$	0.0609	0.0460	0.0596	0.0520	0.0471	0.0482	0.0582	0.0556	0.0423	0.0481	0.0531	0.0423
	$\alpha = 1\%$	0.0068	0.0104	0.0169^{+}	0.0090	0.0082	0.0113	0.0071	0.0070	0.0082	0.0104	0.0113	0.0024*
		TB1	*TB2*	*EW*	*HG*	*TB1*	*TB2*	*EW*	*HG*	*TB1*	*TB2*	*EW*	*HG*
Bonds	$\alpha = 10\%$	0.0933	0.1081	0.1086	0.0895	0.0979	0.1051	0.1052	0.0739*	0.0982	0.0976	0.1055	0.0803*
	$\alpha = 5\%$	0.0438	0.0499	0.0422	0.0498	0.0585	0.0548	0.0632^{+}	0.0356*	0.0529	0.0486	0.0699^{+}	0.0354*
	$\alpha = 1\%$	0.0046	0.0044	0.0071	0.0022*	0.0060	0.0101	0.0108	0.0030*	0.0060	0.0145	0.0112	0.0050

Note: This table shows the failure rate (proportion of observations above the VaR). Stocks are Dow Jones composite 65 average index (DOW) and NIKKEI 225 stock average index (NIK); FXs are foreign exchange rate of U.S. Dollar (US) and German Mark (DM) to British Pound; Bonds are the U.S. 1-year and 2-year treasury constant maturity rate (TB1, TB2 respectively). EW denotes equal-weight portfolio, HG denotes hedge, a zero investment portfolio. $+$ and $*$ denote too risky and too conservative VaR, respectively. The standard errors are calculated according to the approach of Kupiec (1995).

Table 10: Failure rates for multi-period Value-at-Risk forecasts (CC-GARCH)

		One day horizon				Two days horizon				Five days horizon			
		NIK	*DOW*	*EW*	*HG*	*NIK*	*DOW*	*EW*	*HG*	*NIK*	*DOW*	*EW*	*HG*
Stocks	$\alpha = 10\%$	0.1040	0.1066	0.0927	0.1023	0.0925	0.1060	0.0946	0.0913	0.0973	0.1095	0.1006	0.0975
	$\alpha = 5\%$	0.0576	0.0528	0.0469	0.0500	0.0415	0.0519	0.0572	0.0599	0.0623^+	0.0480	0.0445	0.0512
	$\alpha = 1\%$	0.0221^+	0.0169^+	0.0116	0.0197^+	0.0204^+	0.0200^+	0.0126	0.0102	0.0103	0.0179^+	0.0232^+	0.0192^+
		US	*DM*	*EW*	*HG*	*US*	*DM*	*EW*	*HG*	*US*	*DM*	*EW*	*HG*
FXs	$\alpha = 10\%$	0.0906	0.0917	0.1210^+	0.1050	0.1068	0.0976	0.1195^+	0.1023	0.1100	0.1063	0.0974	0.0928
	$\alpha = 5\%$	0.0423	0.0456	0.0594	0.0411	0.0423	0.0554	0.0639^+	0.0553	0.0456	0.0619^+	0.0512	0.0577
	$\alpha = 1\%$	0.0059	0.0196^+	0.0172^+	0.0203^+	0.0212^+	0.0175^+	0.0176^+	0.0165^+	0.0093	0.0101	0.0158^+	0.0204^+
		TB1	*TB2*	*EW*	*HG*	*TB1*	*TB2*	*EW*	*HG*	*TB1*	*TB2*	*EW*	*HG*
Bonds	$\alpha = 10\%$	0.1079	0.0947	0.1007	0.7360*	0.0938	0.1046	0.0983	0.0850*	0.0962	0.1039	0.1030	0.0843*
	$\alpha = 5\%$	0.0542	0.0415	0.0504	0.0321*	0.0405	0.0574	0.0566	0.0335*	0.0471	0.0401	0.0518	0.0378*
	$\alpha = 1\%$	0.0090	0.0021*	0.0028*	0.0019*	0.0016*	0.0204^+	0.0130	0.0025*	0.0042	0.0048	0.0021*	0.0018*

Note: This table shows the failure rate (proportion of observations above the VaR). Stocks are Dow Jones composite 65 average index (DOW) and NIKKEI 225 stock average index (NIK); FXs are foreign exchange rate of U.S. Dollar (US) and German Mark (DM) to British Pound; Bonds are the U.S. 1-year and 2-year treasury constant maturity rate (TB1, TB2 respectively). EW denotes equal-weight portfolio, HG denotes hedge, a zero investment portfolio. $+$ and $*$ denote too risky and too conservative VaR, respectively. The standard errors are calculated according to the approach of Kupiec (1995).

5.3 Expected shortfall

Although value-at-risk conveniently summarizes complex positions in a single figure, it has been argued (Artzner et al., 1997), that it should be complemented by other statistics as it disregards any loss beyond the VaR level (known as the 'tail risk' issue). VaR is also not sub-additive, i.e. the risk of a composite portfolio is less than or equal to the sum of the risk of individual portfolios.

Expected shortfall (ES) has been proposed to alleviate these shortcomings. It is defined as the expected loss conditional on exceeding the VaR, which is given by:

$$ES^{\alpha}_{t:t+h} = E[\left(r_{t:t+h} | r_{t:t+h} \leq -VaR^{\alpha}_{t:t+h}\right) | I_t]. \tag{17}$$

Thus, expected shortfall provides information about the size of the potential losses, given that a loss larger than VaR occurred. This measure has also been shown to be sub-additive (Acerbi and Tasche, 2002).

In this section, we assess our bivariate multifractal model in terms of the performance of multi-period ES forecasts by comparing the empirical and forecasted expected shortfall. As for the VaR forecasts, expected shortfall forecasts based on the Calvet/Fisher/Thompson model and the CC-GARCH(1, 1) model were also computed as benchmarks. The results are reported in Tables 1 to 3. Numbers inside the parentheses are the empirical expected shortfalls, numbers without parentheses are the forecasted ones based on these three models. Bold numbers show those cases for which we cannot reject identity of the empirical and forecasted ES, i.e. the empirical value falling into the range between the 2.5 to 97.5 percent Monte Carlo quantiles of the simulated ones from the particle swarm.

Table 11 shows quite positive results for the new model for stock indices, except for three too risky cases (the simulated ES above the empirical one) at the 1% level. For foreign exchange rates, the ES forecasts give only one too conservative case (the simulated ES below the empirical one) for hedge portfolios at the 1% level of the five-day horizon. For U.S. bonds, we find seven unsuccessful cases among a total of 24 scenarios. The performance of the Calvet/Fisher/Thompson model is provided in Table 2. ES forecasts of stock indices show two excessively risky cases for NIK and DOW at the 1% level at the one and five-day horizons, respectively. The results for foreign exchange rates only have one too conservative HG portfolio at the one-day horizon. For U.S. treasury bond maturity rates, there is a total of eight failures.

For the CC-GARCH(1, 1) model, Table 3 reports successful forecasts for NIK, but shows six failures for the rest of the stock index portfolios. For foreign exchange rates, we find again six failures across all scenarios although forecasts on equal-weight portfolios are successful. Furthermore, results for U.S. treasury bond rates are very disappointing in that all cases at 1% level fail, as well as most forecasts of shortfalls of hedge portfolios.

Again, comparison of expected shortfall shows that the MSM type models provide better forecasts than those derived from CC-GARCH. As with VaR, the performance of the CFM and our alternative approach is head-to-head without any clear tendency for one to perform better than the other.

Table 11: Multi-period expected shortfall forecasts (Liu/Lux model)

		One day horizon				Five days horizon			
		NIK	*DOW*	*EW*	*HG*	*NIK*	*DOW*	*EW*	*HG*
Stocks	$\alpha = 10\%$	(-1.99) **-1.84**	(-2.77) **-2.36**	(-1.78) **-1.64**	(-3.34) **-2.69**	(-1.99) **-1.93**	(-2.77) **-2.38**	(-1.78) **-1.61**	(-3.34) **-2.70**
	$\alpha = 5\%$	(-2.60) **-2.42**	(-3.54) **-3.79**	(-2.28) **-2.03**	(-4.27) **-4.48**	(-2.60) **-2.47**	(-3.54) **-3.21**	(-2.28) **-2.00**	(-4.27) **-4.43**
	$\alpha = 1\%$	(-4.42) **-4.09**	(-5.72) -5.13	(-3.75) **-3.96**	(-6.99) -6.10	(-4.42) **-3.81**	(-5.72) **-4.44**	(-3.75) **-3.35**	(-6.99) -6.07
		US	*DM*	*EW*	*HG*	*US*	*DM*	*EW*	*HG*
FXs	$\alpha = 10\%$	(-1.02) **-1.06**	(-0.92) **-0.91**	(-0.80) **-0.78**	(-1.11) **-1.23**	(-1.02) **-1.07**	(-0.92) **-0.94**	(-0.80) **-0.72**	(-1.11) **-1.24**
	$\alpha = 5\%$	(-1.30) **-1.37**	(-1.17) **-1.17**	(-1.03) **-0.99**	(-1.37) **-1.57**	(-1.30) **-1.40**	(-1.17) **-1.18**	(-1.03) **-1.05**	(-1.37) **-1.59**
	$\alpha = 1\%$	(-2.08) **-2.18**	(-1.83) **-1.87**	(-1.68) **-1.49**	(-2.00) **-2.21**	(-2.08) **-2.27**	(-1.83) **-1.91**	(-1.68) **-1.53**	(-2.00) -2.47
		TB1	*TB2*	*EW*	*HG*	*TB1*	*TB2*	*EW*	*HG*
Bonds	$\alpha = 10\%$	(-0.31) **-0.55**	(-0.49) **-0.74**	(-0.44) **-0.54**	(-0.25) **-0.39**	(-0.31) **-0.50**	(-0.49) **-0.62**	(-0.44) **-0.49**	(-0.25) **-0.44**
	$\alpha = 5\%$	(-0.42) **-0.74**	(-0.67) -1.07	(-0.59) **-0.76**	(-0.38) -0.84	(-0.42) **-0.65**	(-0.67) **-0.77**	(-0.59) **-0.85**	(-0.38) -0.87
	$\alpha = 1\%$	(-0.59) **-0.91**	(-0.83) -1.38	(-0.80) **-0.99**	(-0.58) -1.06	(-0.59) **-0.84**	(-0.83) **-1.07**	(-0.80) -1.29	(-0.58) -1.07

Note: This table reports the expected shortfall (ES) forecast based on the bivariate MF model, numbers in parentheses are the empirical realized ES values. Stocks are Dow Jones composite 65 average index (DOW) and NIKKEI 225 stock average index (NIK), FXs are foreign exchange rate of U.S. Dollar (US) and German Mark (DM) to British Pound, Bonds are the U.S. 1-Year and 2-year treasury constant maturity rate (TB1, TB2 respectively). EW denotes the equal-weight portfolio, HG denotes the hedge portfolio. Numbers inside parentheses give the empirical ES, and numbers outside parentheses are forecasts from the model. Bold numbers show those cases for which we cannot reject identity of the empirical and forecasting ES, i.e. the empirical value falls into the range between the 2.5 to 97.5 percent quantile of the forecasted ones.

Table 12: Multi-period expected shortfall forecasts (Calvet/Fisher/Thompson model)

		One day horizon				Five days horizon			
		NIK	*DOW*	*EW*	*HG*	*NIK*	*DOW*	*EW*	*HG*
Stocks	$\alpha = 10\%$	(-1.99) **-1.81**	(-2.77) **-2.42**	(-1.78) **-1.58**	(-3.34) **-2.98**	(-1.99) **-1.83**	(-2.77) **-2.29**	(-1.78) **-1.53**	(-3.34) **-3.51**
	$\alpha = 5\%$	(-2.60) **-2.39**	(-3.54) **-3.13**	(-2.28) **-1.96**	(-4.27) **-4.31**	(-2.60) **-2.28**	(-3.54) **-3.90**	(-2.28) **-1.90**	(-4.27) **-4.53**
	$\alpha = 1\%$	(-4.42) -3.97	(-5.72) **-5.93**	(-3.75) **-3.88**	(-6.99) **-6.65**	(-4.42) **-4.33**	(-5.72) -5.11	(-3.75) **-2.83**	(-6.99) **-6.76**
		US	*DM*	*EW*	*HG*	*US*	*DM*	*EW*	*HG*
FXs	$\alpha = 10\%$	(-1.02) **-1.19**	(-0.92) **-0.95**	(-0.80) **-0.84**	(-1.11) **-1.38**	(-1.02) **-1.26**	(-0.92) **-0.97**	(-0.80) **-0.87**	(-1.11) **-1.40**
	$\alpha = 5\%$	(-1.30) **-1.55**	(-1.17) **-1.24**	(-1.03) **-1.06**	(-1.37) **-1.55**	(-1.30) **-1.47**	(-1.17) **-1.27**	(-1.03) **-1.12**	(-1.37) **-1.56**
	$\alpha = 1\%$	(-2.08) **-2.31**	(-1.83) **-1.99**	(-1.68) **-1.41**	(-2.00) -2.71	(-2.08) **-2.30**	(-1.83) **-2.09**	(-1.68) **-1.73**	(-2.00) **-2.36**
		TB1	*TB2*	*EW*	*HG*	*TB1*	*TB2*	*EW*	*HG*
Bonds	$\alpha = 10\%$	(-0.31) **-0.52**	(-0.49) **-0.73**	(-0.44) **-0.54**	(-0.25) **-0.46**	(-0.31) **-0.57**	(-0.49) **-0.43**	(-0.44) **-0.37**	(-0.25) **-0.38**
	$\alpha = 5\%$	(-0.42) **-0.73**	(-0.67) **-0.93**	(-0.59) **-0.76**	(-0.38) -0.81	(-0.42) **-0.71**	(-0.67) **-0.92**	(-0.59) **-0.42**	(-0.38) -0.72
	$\alpha = 1\%$	(-0.59) -1.00	(-0.83) -1.34	(-0.80) **-1.07**	(-0.58) -1.05	(-0.59) **-0.84**	(-0.83) -1.37	(-0.80) -1.29	(-0.58) -0.95

Note: This table reports the expected shortfall (ES) forecast based on the bivariate MF model, numbers in parentheses are the empirical realized ES values. Stocks are Dow Jones composite 65 average index (DOW) and NIKKEI 225 stock average index (NIK), FXs are foreign exchange rate of U.S. Dollar (US) and German Mark (DM) to British Pound, Bonds are the U.S. 1-Year and 2-year treasury constant maturity rate (TB1, TB2 respectively). EW denotes the equal-weight portfolio, HG denotes the hedge portfolio. Numbers inside parentheses give the empirical ES, and numbers outside parentheses are the corresponding forecasts from the model. Bold numbers show those cases for which we cannot reject identity of the empirical and forecasting ES, i.e. the empirical value falls into the range between the 2.5 to 97.5 percent quantile of the forecasted ones.

Table 13: Multi-period Expected shortfall forecasts (CC-GARCH model)

		One day horizon				Five days horizon			
		NIK	*DOW*	*EW*	*HG*	*NIK*	*DOW*	*EW*	*HG*
Stocks	$\alpha = 10\%$	(-1.99) **-2.16**	(-2.77) **-2.98**	(-1.78) **-1.49**	(-3.34) **-3.09**	(-1.99) **-1.78**	(-2.77) **-2.79**	(-1.78) **-1.55**	(-3.34) **-2.97**
	$\alpha = 5\%$	(-2.60) **-2.76**	(-3.54) **-3.60**	(-2.28) **-2.32**	(-4.27) **-4.53**	(-2.60) **-2.73**	(-3.54) **-3.85**	(-2.28) **-2.30**	(-4.27) -4.81
	$\alpha = 1\%$	(-4.42) **-4.71**	(-5.72) -6.18	(-3.75) -3.27	(-6.99) -6.50	(-4.42) **-4.05**	(-5.72) **-5.37**	(-3.75) -3.22	(-6.99) -6.07
		US	*DM*	*EW*	*HG*	*US*	*DM*	*EW*	*HG*
FXs	$\alpha = 10\%$	(-1.02) **-0.89**	(-0.92) **-0.87**	(-0.80) **-1.12**	(-1.11) **-1.39**	(-1.02) **-1.37**	(-0.92) **-0.96**	(-0.80) **-0.77**	(-1.11) **-1.25**
	$\alpha = 5\%$	(-1.30) **-1.45**	(-1.17) **-1.02**	(-1.03) **-1.43**	(-1.37) **-1.71**	(-1.30) **-1.61**	(-1.17) **-1.42**	(-1.03) **-1.20**	(-1.37) -1.79
	$\alpha = 1\%$	(-2.08) **-1.83**	(-1.83) -1.40	(-1.68) **-2.00**	(-2.00) -2.52	(-2.08) -2.57	(-1.83) -2.29	(-1.68) **-1.95**	(-2.00) -2.47
		TB1	*TB2*	*EW*	*HG*	*TB1*	*TB2*	*EW*	*HG*
Bonds	$\alpha = 10\%$	(-0.31) **-0.43**	(-0.49) **-0.67**	(-0.44) **-0.49**	(-0.25) -0.67	(-0.31) **-0.32**	(-0.49) **-0.32**	(-0.44) **-0.54**	(-0.25) -0.63
	$\alpha = 5\%$	(-0.42) **-0.69**	(-0.67) **-0.92**	(-0.59) **-0.70**	(-0.38) -0.82	(-0.42) **-0.70**	(-0.67) **-0.89**	(-0.59) **-0.81**	(-0.38) -0.85
	$\alpha = 1\%$	(-0.59) -0.107	(-0.83) -1.41	(-0.80) -1.45	(-0.58) -1.01	(-0.59) -1.05	(-0.83) -1.26	(-0.80) -1.29	(-0.58) -0.97

Note: This table reports the expected shortfall (ES) forecast based on the bivariate MF model, numbers in parentheses are the empirical realized ES values. Stocks are Dow Jones composite 65 average index (DOW) and NIKKEI 225 stock average index (NIK), FXs are foreign exchange rate of U.S. Dollar (US) and German Mark (DM) to British Pound, Bonds are the U.S. 1-Year and 2-year treasury constant maturity rate (TB1, TB2 respectively). EW denotes the equal-weight portfolio, HG denotes the hedge portfolio. Numbers inside parentheses give the empirical ES, and numbers outside parentheses are the corresponding forecasts from the model. Bold numbers show those cases for which we cannot reject identity of the empirical and forecasting ES, i.e. the empirical value falls into the range between the 2.5 to 97.5 percent quantile of the forecasted ones.

6 Concluding remarks

In this paper, we have introduced a new parsimonious bivariate multifractal model. We have also implemented estimation of its parameters via maximum likelihood and simulation based inference. To shed light on the applicability of this new model, two well-known instruments in financial risk management have been employed in comparison with both the alternative bivariate MF model of CFM and a more traditional multivariate GARCH process. We have considered financial time series of stock markets, foreign exchange rates and bond markets in our empirical application. As it turned out multifractal models provide promising results throughout for both value-at-risk and expected shortfall forecasts. There was little difference in the performance of our new model and the one proposed by Calvet, Fisher, Thompson (2000). Since there are fewer parameters to be estimated in our approach, these findings speak of some redundancy in the number of model parameters (e.g., the use of a bivariate Binomial distribution with different parameters for the multipliers in CFM). As in previous research (Lux, 2008, Liu et al., 2009), we find that the results from applications of multifractal models are quite robust across specifications.

A few avenues suggest themselves for further research: although our heuristic approach for model selection seemed to work quite satisfactorily, one would certainly like to develop more rigorous specification tests. Furthermore, extensions of the present bivariate approach to higher-dimensional portfolios would be welcome for practical applications. Because of the increasing computational demands, we probably would have to give up likelihood-based methods of inference and turn to less demanding moment estimators for the parameters of the multifractal models.

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