Adapting long-lived investments under climate change uncertainty☆

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ABSTRACT

Does climate change adaptation require that investments are designed to be more robust? What about when climate change is more uncertain? What if the climate changes faster? This decision problem is difficult if the design of the investments is irreversible for their lifetime, for instance, in the construction industry. We study an irreversible design decision when the investment starts, combined with an irreversible option to abandon. The design determines the investment's robustness to sustain detrimental conditions. We find that for short-lived investments, optimal robustness decreases if the climate changes faster, and increases if uncertainty is higher. For long-lived investments, these effects reverse. This has implications for decision makers who plan infrastructure adaptation, for instance, that adverse climate change does not require more robust investments under the identified circumstances.

1. Introduction

Suppose a real estate investor is considering the construction of a new building, but expects hydro-meteorological risk from climate change. What would be the best design of the building when facing such a risk? It might make sense to adapt to climate change by choosing a more robust design, that is, one that can sustain more detrimental climatic conditions in a better way. Another option would be to choose a less robust design, optimized for the present climate, while anticipating that the building might have a shorter economic lifetime.

This problem structure applies to many investment decisions involving adaptation to climate change, be they private or public. It has recently been considered, for instance, that common chattel houses in Barbados, which are vulnerable to more frequent hurricanes, should be replaced by more permanent concrete homes instead (Barbados Today, 2021). Since the precise path of climate change is uncertain *ex ante*, developers face the question: Should designs of those houses indeed be more robust to novel climate conditions, so that they suffer less damage from more frequent hurricanes, or should they remain less robust, therefore reducing their expected lifetime, but being committed to possibly wrong projections of hurricane frequency only for a shorter time? More generally, adaptation investment in projects such as power plants, chemical and steel factories, buildings, airports, dams, and mega-projects are frequently characterized by technology commitment. Adaptation by retrofitting these investments to a new climate (e.g. by replacing construction materials) can be quite costly, so much so that initial decisions about their design are de facto irreversible over the course of their economic lifetime (e.g. Turvey, 2000; Flyvbjerg, 2014; Ansar et al., 2014). Over these kinds of long time scales, the benefit streams of the investments (partially driven by climate change) are (i) not constant and (ii) quite difficult to predict. Our paper shows when a less robust design pays off for such decision problems.

Climate dependence of benefit streams plays out, in particular, for adapting infrastructures to climate change, an issue which has received increasing attention in recent years (e.g., OECD, 2008; UNEP, 2016; Roggero et al., 2018). It is generally beneficial to fit technical designs (e.g., type of concrete, steel, or machinery) to climatic conditions (e.g., temperature, precipitation, wind patterns), since maintenance will then be cheaper, durability will be increased, and service disruptions will be less likely (cf. IPCC, 2022). It is thus reasonable to adapt the design of infrastructures. This may cost up to $100b annually (or even more, depending on the estimate, Stern, 2007; OECD, 2008; UNEP, 2016), or $70b annually for coastal protection alone (Hinkel et al., 2014), with huge variation depending on the assumed climate change scenario (Lincke and Hinkel, 2018). What do uncertainties from climate change imply, then, for the design of such irreversible investments? The conceptual novelty of our paper lies in jointly considering two effects here: the effect of the initial and irreversible design decision on the lifetime of an investment (which also depends on how uncertainty unfolds), and the effect of the anticipated expected lifetime on the initial design decision.

We analyze these questions by generally determining the optimal lifetime and design of an investment in the presence of a stochastic process. The decision problem combines a real option to abandon with an irreversible decision about design parameters at the beginning of the investment's lifetime. The design, interpreted as robustness, makes the investment's benefit less sensitive to the state variable of the stochastic process (climate change). Although this seems like a common real-world decision problem, we are not aware of solutions in the vast real options literature (cf. Trigeorgis and Tsekrekos, 2018). This might be due to a seemingly intuitive argument: With higher uncertainty, it seems that the marginal benefit from robustness would be larger, such that a more robust design would pay. Paradoxically, we find that this is not necessarily the case.

There are second-order effects resulting from robustness and lifetime being interdependent. If projects are comparatively long-lived, higher uncertainty might be so detrimental that it is not worthwhile to choose a more robust design. Our findings are thus important for decision makers who plan adaptation in order to avoid inefficient expenditures. Possibly counter-intuitive to practice, they need to consider that climate change adaptation of long-lived-investments needs less robustness in some cases.

In the literature, technology commitment has been studied in the context of the investment-uncertainty relationship (e.g., Ramey and Ramey, 1995; Sarkar, 2000; Jovanovic, 2006). This literature generally takes a macro-perspective and focuses on the option to invest instead of the option to abandon. Although an investment's lifetime or time to abandon has been analyzed (Farzin et al., 1998; Dahlgren and Leung, 2015), these studies do not include an irreversible design decision. Some papers have addressed adaptation of investments to climate change in a way related to ours (e.g., Fisher and Rubio, 1997; Callaway, 2004; Hallegatte, 2009; de Bruin and Ansink, 2011; Felgenhauer and Webster, 2014; vander Pol et al., 2014), but most of them either do not explicitly focus on irreversible design and abandonment, or take a less formal approach. There are two challenges here (cf. Hallegatte, 2009): (i) If climate change is subject to an ongoing trend, the design of long-lived infrastructures needs to fit a broader range of climatic conditions. (ii) Projections about the rate of climate change are prone to different kinds of uncertainty (e.g., Weitzman, 2013; Heal and Millner, 2014; IPCC, 2022).

There is also some literature on adjustment costs of production capacity, inventories or land development in the presence of uncertainty (e.g., Arrow and Fisher, 1974; Abel and Eberly, 1994; Van Mieghem, 2003; Ye and Duenyas, 2007). Some of those papers (see also Sundaresan and Wang, 2007) study irreversible investment timing problems in combination with the possibility of a later capacity expansion, or a further decision variable (e.g. another production factor) which is completely flexible. Instead, we consider a situation where the decision to continue or abandon the investment is the only kind of flexibility (then, Myers and Maid (2004) show that the option to abandon does not admit closed-form solutions in general; see also Lavrutich (2017) for a different set-up). We thus study a case which contrasts previous findings in terms of flexibility.

The additional twist of our paper is, however, that the investor can anticipate the effect of her initial and irreversible design decisions on the own (expected) future decisions to continue or abandon, and takes these into account when designing the investment. We think that this twist is quite important, since rational decision-makers will have an interest to coherently plan design and lifetime of an investment. Surprisingly, we are not aware that this set-up has been addressed in the literature. Our paper thus contributes by investigating the interdependence of two irreversible decisions: design and abandonment. We show precisely how stronger climate change trends and higher uncertainty affect optimal investment lifetimes in different ways, and derive conditions that can be used to differentiate the respective outcomes.

Section 2 presents the model's decision structure and our general results. Section 3 uses the model to investigate a particularly insightful optimal stochastic dynamic control problem to maximize an investment's expected net value. Analytical comparative statics results for different exogenous variables, in particular uncertainty, are presented in Section 4, followed by numerical experiments. The proofs are relegated to Appendix. Section 6 concludes.

2. General model setup and decision structure

We analyze the decision on a long-lived investment that operates within dynamic and uncertain climate conditions x, modeled by a geometric stochastic process

$$dx = \mu x \, dt + \sigma x \, dz, \tag{1}$$

with $x(0) = x_0 > 0$, trend parameter $\mu > 0$, standard deviation $\sigma > 0$, and (z_t) being a standardized Wiener Process, so that $x \geq 0$. We will call σ uncertainty in this paper. The climate conditions influence the investment's stream of current benefits over time t, so that the decision's objective is to maximize

$$J(x_0, a, \mu, \sigma) = E\left[\int_0^{T^\star} \pi(x, a)e^{-rt}dt\right] - C(a), \tag{2}$$

with respect to a technical design vector or parameter a that describes the investment's properties, and with respect to the time $T^\star$ where the investment is ultimately stopped. Here and in the following, $E[\cdot]$ is the expectation operator, and $\pi(x, a)$ denotes the current benefits, which depend on how the design a fits the climate conditions x at each specific point in time. Thus, the current benefit stream changes over time. Current benefits might represent, for instance, the benefits of using a safe building, the profits of running a power plant, or the social gains from a reliable road or railway line. If current benefits π are a decreasing function in x (the case we have in mind here), we can speak of the climate conditions being detrimental: They cause damage, being the difference between benefits for x and for x_0. Then, if $\partial_a \pi \geq 0, \partial_{a,x} \pi > 0$, we can interpret the design a as robustness (here and in the following, ∂. denotes partial derivatives). First, a does not increase damage, second a damage-reducing effect becomes larger for a more detrimental climate. Note that the considered climate conditions x can be the average temperature, but can actually be any appropriate climate-related metric which enters the current benefits function, e.g. precipitation, river run-off, heating degree days, power plant outages (see Steinhäuser and Eisenack, 2020, for instance), and so on. It is only required that it can be analyzed as a geometric Brownian motion. A robust design can be considered, for instance, as concrete which is adapted to higher temperatures, reservoirs which store more water, road material which sustains more frequent frost-thaw cycles, or power plant technology which is less prone to cooling water shortage. Yet, we keep the analysis more general in the remainder of this Section by not making such assumptions (and interpretations). The subsequent Sections will be devoted to the case with detrimental climate change and design as robustness.

Current benefits are discounted to present value at rate $r > 0$. We assume technology commitment, that is, the technical design is fixed over the complete investment's lifetime. The investment costs $C(a)$ with $C' > 0, C'' \geq 0$ depend on the design and incur at the start. This kind of irreversibility can be justified, for example, if the costs of a retrofitting the investment to new conditions are prohibitively costly. For instance, shifting from wooden to concrete buildings practically requires a complete re-construction. This also applies when the rail type used for a rail track is replaced by other types in order to reduce traffic disruptions during heat waves. A further example are drainage pipes (e.g. Moore et al., 2016; Ngamalieu-Nengoue et al., 2019). If they are upsized to better deal with flooding, then expensive underground construction (actually a re-building), is needed.

After the investment is constructed based on the chosen design, the remaining decision is when its lifetime should end. We thus assume the following two-stage decision structure:

> First stage: The irreversible design a is chosen.
> Second stage: At each point in time, a decision is made whether to continue or stop the investment. Stopping at some time $T^\star$ is an irreversible decision.

This problem will be solved by backward induction, where the second stage is a standard stopping problem. At the time when the investment starts, we do not know the stopping time yet, but we can, in principle, determine the expected stopping time $E[T^\star](x_0, a, \mu, \sigma)$, which depends in particular on the design a as chosen in the first stage. In the first stage, the design decision regarding a will depend, in turn, on the stopping time anticipated for the second stage. We aim at determining the optimal design $a^\star$, which yields the expected lifetime with optimal design $T^{\star\star} = E[T^\star](x_0, a^\star, \mu, \sigma)$. We are interested in the comparative statics of the optima, in particular with respect to (μ, σ), to see whether the optimal design or lifetime will be extended or shortened if the climate is changing with a stronger trend, or if climate change is more uncertain.

2.1. Optimal stopping with arbitrary design (second stage)

First assume that a has an arbitrary value and concentrate on optimal stopping for this given design. Some general implications can be drawn from the decision structure without specifying the functional form of current benefits $\pi(x, a)$, assuming that the form at least leads to a well-posed problem. In the second stage, stopping problems of our kind typically yield a decision rule with a cutoff value $x^\star(a, \mu, \sigma) > x_0$. At the optimal stopping time $T^\star(x_0, a, \mu, \sigma)$ we have $x(T^\star) = x^\star(a, \mu, \sigma)$ (cf. Dixit and Pindyck, 1994). Differentiation yields:

Proposition 1. *If a cutoff value $x^\star > x_0$ exists and $\mu > \frac{1}{2}\sigma^2$, then the second stage stopping problem has the following comparative statics properties:*

$$\frac{dE[T^\star]}{d\mu} = \frac{1}{\mu - \frac{1}{2}\sigma^2}\Big(\frac{\partial_\mu x^\star}{x^\star} - E[T^\star]\Big), \tag{3}$$

$$\frac{dE[T^\star]}{d\sigma^2} = \frac{1}{\mu - \frac{1}{2}\sigma^2}\Big(\frac{\partial_\sigma x^\star}{x^\star} + \frac{1}{2}E[T^*]\Big), \tag{4}$$

$$\frac{dE[T^\star]}{da} = \frac{1}{\mu - \frac{1}{2}\sigma^2}\frac{\partial_a x^\star}{x^\star}. \tag{5}$$

This is straightforward to determine, since the expected stopping time of a geometric Brownian motion follows $E[T^\star] = \frac{1}{\mu - \frac{1}{2}\sigma^2}\log\frac{x^\star}{x_0}$ (e.g., Wilmott et al., 1993, AppendixB.1). The inequality for μ, σ is needed throughout the paper, since the problem is not well-posed otherwise: uncertainty is so high that the investment would never be stopped because there is always a sufficient probability that the climate conditions will become beneficial again. Eq. (5) shows that a higher design parameter a shifts the expected stopping time in the same direction as it shifts the cutoff value, which is quite intuitive. The effect of a stronger trend

(with a given design), Eq. (3), is more complicated. One might expect that, for example, if μ is larger, the expected time at which the investment's design will no longer fit the climate conditions well can be expected to come earlier. This is indeed the case if the cutoff value $x^\star$ decreases for a stronger trend (as will be the case in the next Section). But the latter case does not hold in general. All we can say here is that stopping earlier in light of a stronger trend is more likely if the investment is expected to have a longer lifetime anyway. Higher uncertainty with a given design, Eq. (4), necessarily leads to a longer expected lifetime if it also raises the cutoff value (as will be discussed in the next section). If the cutoff value becomes smaller, this opposite effect is more likely compensated for with particularly long-lived investments.

2.2. Optimal design (first stage)

In the first stage, however, the design a is not fixed, but selected to maximize J. Choosing the design implicitly determines the cutoff value. Usually, the cutoff value can be determined from a value function $h(x)$ that expresses the expected value of not stopping the investment (yet) for given conditions x and assuming that the investment is stopped at the optimal time in the future (cf. Dixit and Pindyck, 1994). Thus, $J(x_0, a, \mu, \sigma) = h(x_0) - C(a)$ (if the project is not immediately stopped, that is, if $h(x_0) \leq 0$), and the first-order and second-order conditions are

$$\partial_a J(x_0, a, \mu, \sigma) = \frac{d}{da} h(x_0) - C'(a) = 0, \tag{6}$$

$$\partial_{aa} J(x_0, a, \mu, \sigma) = \frac{d^2}{da^2} h(x_0) - C''(a) < 0. \tag{7}$$

Now suppose that Eq. (7) holds. Solving Eq. (6) for a yields the optimal design $a^\star(x_0, \mu, \sigma)$, and thus the optimally designed expected lifetime $T^{\star\star}(x_0, \mu, \sigma) = E[T^\star](x_0, a^\star(x_0, \mu, \sigma), \mu, \sigma)$. By making use of Eq. (3)-Eq. (5), these functions have the following comparative statics properties (with $\doteq$ denoting equivalence in signs):

Proposition 2. *If the investment is optimally stopped according to Proposition 1 in the second stage, and an optimal design $a^\star$ exists in the first stage, the solution has the following comparative statics properties:*

$$\partial_\mu a^\star(x_0, \mu, \sigma) = -\frac{\partial_{a\mu} J}{\partial_{aa} J} \doteq \frac{d^2}{da\,d\mu} h(x_0), \tag{8}$$

$$\partial_\sigma a^\star(x_0, \mu, \sigma) = -\frac{\partial_{a\sigma} J}{\partial_{aa} J} \doteq \frac{d^2}{da\,d\sigma} h(x_0), \tag{9}$$

$$\partial_\mu T^{\star\star}(x_0, \mu, \sigma) = \partial_\mu E[T^\star] + \partial_a E[T^\star] \cdot \partial_\mu a^\star =$$
$$= \frac{1}{\mu - \frac{1}{2}\sigma^2} \left(\frac{\partial_\mu x^\star}{x^\star} - T^{\star\star} + \frac{\partial_a x^\star}{x^\star} \partial_\mu a^\star \right), \tag{10}$$

$$\partial_{\sigma^2} T^{\star\star}(x_0, \mu, \sigma) = \partial_{\sigma^2} E[T^\star] + \partial_a E[T^\star] \cdot \partial_{\sigma^2} a^\star =$$
$$= \frac{1}{\mu - \frac{1}{2}\sigma^2} \left(\frac{\partial_{\sigma^2} x^\star}{x^\star} + \frac{1}{2} T^{\star\star} + \frac{\partial_a x^\star}{x^\star} \partial_{\sigma^2} a^\star \right). \tag{11}$$

The first two equations show that the effects of the trend and uncertainty depend on the marginal value of the design. If a stronger trend or higher uncertainty enhances the marginal value of design, it is optimal to choose a larger design parameter.

The last two equations show that the effect for the optimal expected lifetime can be decomposed into two effects. The first summand in the shorter versions of Eq. (10), Eq. (11) is the direct effect (as already discussed for the second stage). The further summands refer to an indirect effect. In Eq. (10) one might expect, for example, that an investment that is more robust to climate change (higher a) will be stopped at a later time. If the trend is stronger, and suppose that the direct effect leads to a shorter lifetime, fewer benefits during the expected lifetime are likely. In this case, robustness does not pay off as much, and a less robust design might be optimal. Then, the expected lifetime would consistently be shorter. In such a case, policy makers or investors who plan adaptation to climate change and have reasons to update their estimated μ upwards should chose more short-lived projects.

However, it is not obvious at this point whether it might also be optimal to make the investment more robust to compensate a stronger climate change trend ($\partial_\mu a^\star > 0$). If this were the case, the direct and indirect effects for $\partial_\mu T^{\star\star}$ would point in opposite directions. What we can say for certain, from the expanded version of Eq. (10), is that the lifetime of a particularly long-lived investment is more likely to be shortened in the presence of a stronger trend.

A similar argument can be made for higher uncertainty in Eq. (11). The direct effect describes the impact of uncertainty on the expected stopping time (which might be expected to be positive due to a higher premium to wait with stopping). The indirect effect $\partial_a E[T^\star] \cdot \partial_{\sigma^2} a^\star$ is driven by whether or not higher uncertainty incentivizes a more robust design. Since it cannot be generally inferred that the direct and the indirect effect point in the same direction, results depend on the further specification of the optimization problem. However, the expanded version of Eq. (11) shows that particularly long-lived investments more likely have an extended lifetime in presence of higher uncertainty. If adaptation decision makers become more uncertain about how climate change enfolds, they should plan that investments who reach quite far into the future anyway (e.g. bridges, dykes or dams[1]) live even longer.

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[1] Dykes and dams do protect investments, and could thus be considered as part of the investments' design.

3. A model with optimal stopping and robustness

As several comparative statics results above depend on the detailed problem specification, this section analyzes the model for an ideal-type case with linear current benefits function and one design parameter. This specification is an insightful case: We will see below that it already admits that rising uncertainty can make either more or less robust designs optimal (depending on the lifetime). It is the most clear-cut representation of all settings where the climate conditions are detrimental for the benefit stream, while this design can be interpreted as robustness. For non-linear cases, our findings consequently imply that all equations in Proposition 2 can be positive or negative as well, yet in an even more complicated way. The specification has the additional advantage that central results can be derived analytically, and that it can be applied in multiple ways. Many climate change damage functions, e.g. for rail lines, bridges or electricity supply, can be well approximated by linear regression (Neumann et al., 2020). Linear aggregate damage functions are also common in the literature (e.g. Pavlova and de Zeeuw, 2012; Hagen and Eisenack, 2019). As another example, economic costs of conventional power plant outages due to heat-waves rise broadly linear in the amount of curtailed capacity and the frequency of outages (Pechan and Eisenack, 2014; Eisenack, 2016). Note that the interpretation of x and a also admits some generalization, e.g., by re-scaling the stochastic process. More complicated specifications likely require either numerical analysis or a piecewise linear approximation with our model.

We first solve the optimal stopping problem and delegate the comparative statics to the subsequent section. The conditions x and the design parameter $a > 0$ are assumed to determine the investment's current benefit according to $\pi(x, a) = \gamma - x/a$ with some $\gamma > 0$. Therefore, we model detrimental climate change: A higher value for μ implies a more negative trend for current benefit. The benefit π is always below a maximum γ, and diminishes to zero if the climate x approaches γa. Since $\partial_a \pi \geq 0, \partial_{a,x} \pi > 0$, we can conceive the design parameter as the investment's robustness. We further assume that robustness comes at constant unit costs $c > 0$, so that $C = ca$.

3.1. Optimal stopping with arbitrary design (second stage)

We first study the second stage in which the design a is fixed at some arbitrary level, that is, the decision problem Eq. (2) subject to Eq. (1). This is an autonomous optimal stopping problem in current-value formulation. We can solve this problem by determining the value function $h(x)$ that is required to satisfy the Hamilton–Jacobi–Bellman equation

$$-rh + (\gamma - \frac{x}{a}) + \mu x \, h' + \frac{1}{2}\sigma^2 x^2 h'' = 0. \tag{12}$$

The optimal stopping rule is to continue operation as long as $x(t) < x^\star$, the latter being the cutoff value. At the stopping time $x(T^\star) = x^\star$. The cutoff value is characterized by the standard value matching and smooth pasting conditions $h(x^\star) = 0, h'(x^\star) = 0$. The Appendix shows the following solution:

Proposition 3. *Define the reappearing terms* $\omega := \frac{r-\mu}{r} \frac{\beta}{\beta-1}$ *and* $a_0 := \frac{x_0}{\gamma\omega} > 0$. *If* $a > a_0$, *then the optimal stopping problem Eq. (1), Eq. (2) with* $\pi = \gamma - x/a$ *is solved by the value function*

$$h(x) = \frac{\gamma}{r(\beta-1)}(\frac{x}{x^\star})^\beta - \frac{1}{a(r-\mu)} x + \frac{\gamma}{r}, \tag{13}$$

where $\beta > 0$ *is the positive root of the characteristic polynomial*

$$\frac{1}{2}\sigma^2\beta^2 + (\mu - \frac{1}{2}\sigma^2)\beta - r = 0. \tag{14}$$

It holds that $\omega > 1$, *and the cutoff value* $x^\star = \omega\gamma a$ *respects*

$$\frac{x^\star}{x_0} = \frac{a}{a_0}. \tag{15}$$

The parameter a_0 characterizes an extreme design such that, if $a = a_0$, the initial value is $h(x_0) = 0$: The robustness is so low that the investment would be stopped immediately. A given design $a > a_0$ guarantees that $x^\star > x_0$. The inequality $\omega > 1$ holds because it can be verified that, although both $r \lessgtr \mu$ is possible,[2]

the root β is always between 1 and r/μ. (16)

It follows that $x^\star > \gamma a > 0$, such that the current benefit $\pi(x^\star, a)$ is always negative when the investment is stopped at $t = T^\star$. This is due to the option value of postponing to stop the investment.

3.2. Optimal design (first stage)

We now determine the optimal design $a^\star$: what is the best level of robustness to choose? This is not straightforward, as the net value J does not has a simple shape (see Fig. 1 for an example). The first-order condition Eq. (6) can be re-arranged to $c =$

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[2] Note that $r < \mu$ is, in contrast to other optimal stopping problems in the literature, a reasonable case. Since π is decreasing in x, there are no problems with a non-existing net value J. Both higher r and μ lead to less benefits in the future. The cases $\beta \lessgtr 1$ distinguish whether discounting or the trend dominate in the long run.
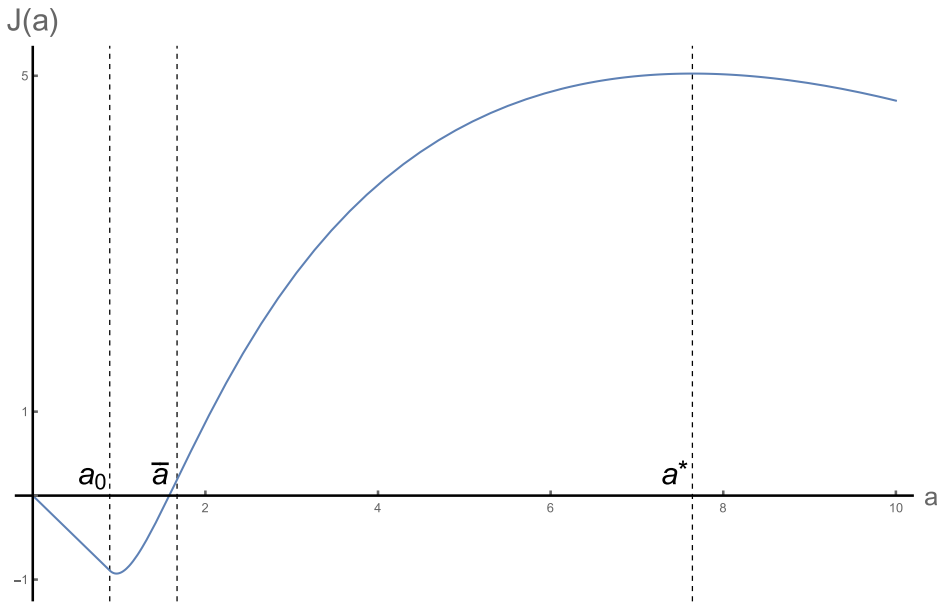
Fig. 1. Numerical example for net value J depending on design a (with $r = 0.02, \mu = 0.09, \sigma^2 = 0.02, c = \gamma = x_0 = 1.00$, thus $a_0 = 0.89, \bar{a} = 1.67, a^\star = 7.64$).

$\frac{x_0}{r-\mu}\left(a^{-2} - (\frac{x_0}{\gamma\omega})^{\beta-1}a^{-\beta-1}\right)$. Generally, this equation cannot be solved for a with an explicit expression. Also the second-order condition is not easy to confirm. Some other features of J can be collected more easily. If $a = a_0$ the net value J is always negative. This follows from the value-matching condition and the definition of a_0 according to $J(x_0, a_0, \mu, \sigma) = h(x_0) - ca_0 = h(x^\star) - ca_0 = -ca_0 < 0$. It further follows from the smooth pasting condition that $J_a(x_0, a_0, \mu, \sigma) < 0$. The following proposition provides sufficient conditions for the existence of an optimum, and a general characterization of its level (see Appendix for the proof and further details). Part of the proof is to show that $J = h(x_0) - ca$ has an inflection point at $a = \bar{a} := (\frac{2}{\beta+1})^{\frac{1}{1-\beta}} a_0 > a_0$.

Proposition 4. *There exists a unique global inner maximum of J,*

$$a^\star = \left(\frac{\beta+1}{2}\right)^{\frac{z}{\beta-1}} a_0, \tag{17}$$

if and only if $c < \phi$ and $J(x_0, a^\star, \mu, \sigma) > 0$. The parameters $z > 1, \phi$ are given in the proof, the former by an implicit equation. Both parameters depend on all model parameters $r, \mu, \sigma, \gamma, c, x_0$ in a non-linear way.

The condition $c < \phi$ expresses an upper limit for the unit costs of robustness. If robustness were more expensive, then the investment would yield a negative net value even if robustness were optimally chosen (such that the investment would not be started at all). This upper limit is actually the marginal value $\frac{d}{da}h(x_0)$ at the inflection point $\bar{a}$. The proof shows that if $a^\star$ exists, J has exactly one minimum and one maximum. The minimum is between a_0 and the inflection point $\bar{a}$, the maximum to the right of the inflection point. While the condition $J(x_0, a^\star, \mu, \sigma) > 0$ seems obvious, it also excludes a corner solution $a = a_0$. Note that Eq. (15) then automatically provides an expression for the cutoff value if robustness is optimal.

The maximum $a^\star$ is the optimal robustness in the presence of uncertainty and the trend, anticipating the expected stopping time at the beginning of the investment. The investment's net value $J(a)$ increases in a if robustness is low since the value of not stopping the investment outweighs the robustness costs. If robustness becomes too large, the marginal gains of robustness become too low in comparison to the marginal costs. More intuition will be provided through the comparative statics and numerical examples in the following section.

3.3. Notes on the option value

The option value and the interpretation of the value function Eq. (13) in the second stage can be further studied by comparing these with the solution that maximizes the net value in the absence of uncertainty, such that $x(t) = x_0 e^{\mu t}$ (see Appendix):

Proposition 5. *For $\sigma = 0$ and $a > \frac{x_0}{\gamma}$, the second stage decision problem is solved by the value function*

$$h^\circ(x) = \frac{\mu\gamma}{r(r-\mu)}(\frac{x}{x^\circ})^{\frac{r}{\mu}} - \frac{1}{a(r-\mu)}x + \frac{\gamma}{r} \tag{18}$$

with cutoff value $x^\circ = \gamma a > 0$.

Obviously, when the investment is stopped at $x = x°$, the current benefit π is exactly zero. There is no gain from further operating the investment, and also no option value. Since $x° = \gamma a < x^\star$, uncertainty leads to stopping the investment at a later time. The role of β in Proposition 3 is taken over by r/μ in Proposition 5. Now consider the difference between the value functions for both cases, that is, the option value

$$\Theta(x) = h(x) - h°(x) = \frac{\gamma}{r} \left(\frac{1}{\beta - 1} (\frac{x}{x^\star})^\beta - \frac{\mu}{r - \mu} (\frac{x}{x°})^{\frac{r}{\mu}} \right). \tag{19}$$

The only difference between the value functions $h, h°$ is their first term. The second and third terms in both value functions represent the value of the investment if it were never stopped, whereas the first terms represent the gain from stopping at the best time. The option value depends on both μ, σ in a non-linear way. We can conclude that $\Theta(x°)$ is positive due to Eq. (16). The effect of the design a on the option value is completely captured by its influence on the cutoff values $x° = \gamma a$ and $x^\star = \omega\gamma a$, so that the parameter ω captures the effects of uncertainty.

4. Comparative statics

We want to know how the optimal expected lifetime and the optimal design depend on various parameters. In particular, do the robustness and the lifetime increase or decrease if climate change is faster or more uncertain? We thus determine the direct and indirect effects in Eq. (10), Eq. (11). We focus on the comparative statics of the stopping problem (second stage) first, and then proceed with the analysis of optimal design (first stage).

4.1. Comparative statics of optimal stopping with arbitrary design (second stage)

The comparative statics of the stopping time with respect to uncertainty and robustness can be determined by using the cutoff value $x^\star$ from Proposition 3 in Proposition 1:

Proposition 6. *Let $x^\star$ and $E[T^\star]$ be the solution to the optimal stopping problem from Proposition 3. Then,*

$$\partial_\mu x^\star < 0, \ \partial_{\sigma^2} x^\star > 0, \ \partial_a x^\star = \gamma\omega > 0, \tag{20}$$

and

$$\partial_\mu E[T^\star] < 0, \ \partial_{\sigma^2} E[T^\star] > 0, \ \partial_a E[T^\star] > 0, \ \partial_{x_0} E[T^\star] < 0. \tag{21}$$

If the trend is stronger, the current benefits deteriorate earlier. This ultimately leads to negative current benefits, and a lower cutoff value according to Eq. (20): the gains from stopping the investment become larger, so this is expected at an earlier time due do to Eq. (21). Higher uncertainty means that more information appears over time. As more information eases the stopping decision, the premium for waiting to stop the investment rises. Thus, Eq. (20) shows the cutoff value becomes larger, that is, less favorable climatic conditions are accepted, and the expected stopping time is later as shown in Eq. (21). Finally, rising robustness makes the investment more beneficial in the light of climate change. Intuitively, a higher cutoff level and later stopping time result.

It helps us to interpret the interlinked effects of a stronger trend and higher uncertainty if we further inspect the option value Θ and how it depends on the design. First observe that $\partial_a x^\star > \gamma = \partial_a x° > 0$ by Propositions 6 and 5. Thus, uncertainty increases the positive effect of robustness on the expected stopping time in Proposition 3. The effect of robustness on the option value, however, is ambiguous (see Appendix):

Proposition 7. *Let $x^\star$ and $x°$ be the solution to the optimal stopping problems from Propositions 3 and 5. Then, $\partial_a\Theta > 0$ if and only if*

$$\gamma a < x°\omega^{\frac{\mu(\beta - 1)}{r - \mu\beta}}. \tag{22}$$

Due to Eq. (16), the exponent is always positive. The inequality shows that increasing robustness raises the option value up to a maximum. For even more robustness, the value of the stopping option is decreasing again. It also shows that robustness raises the option value if the climate becomes increasingly unfavorable.

4.2. Comparative statics of optimal design (first stage)

Now turn to the comparative statics if the irreversible design is optimally chosen. By applying the general Eq. (8) and Eq. (9) to our robustness model, the results are as follows (see Appendix):

Proposition 8. *Assume that the conditions of Proposition 4 hold. Then, $\partial_c a^\star < 0$. For changing uncertainty, $\partial_{\sigma^2} a^\star > 0$ if and only if $T^{\star\star} < \bar{T}_\sigma$, with $\bar{T}_\sigma = (\beta(\mu - \frac{1}{2}\sigma^2))^{-1}$. For a changing trend, $\partial_\mu a^\star > 0$ if and only if $T^{\star\star} > \bar{T}_\mu$, where $\bar{T}_\mu > 0$ is given by a unique solution of an implicit equation in the proof. Both $\bar{T}_\sigma, \bar{T}_\mu$ only depend on μ, σ, r.*
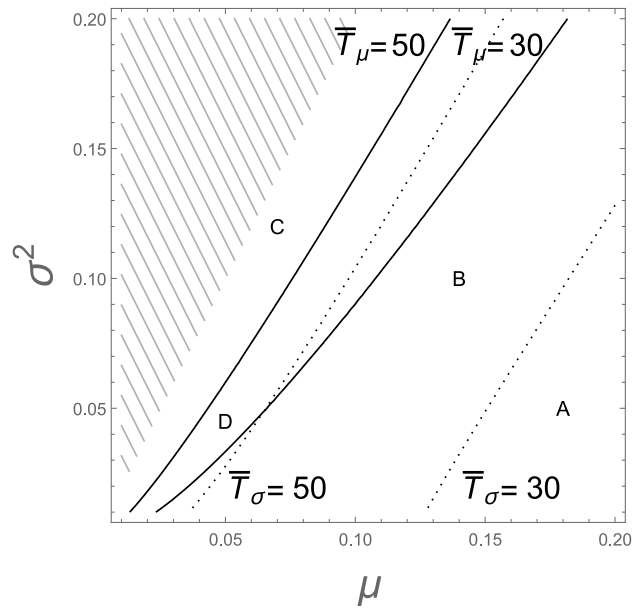
Fig. 2. Thresholds (with $r = 0.05$; solid: isolines for $\bar{T}_\mu$, dotted: isolines for $\bar{T}_\sigma$; for A–D, see text). The shaded area represents cases where the decision problem is not well-posed according to Proposition 1.

This highlights that the effect of uncertainty and the trend on the optimal design is not a simple one. While a stronger trend or higher uncertainty might intuitively imply a more robust design in some cases, it might not be worthwhile in other cases. Uncertainty or the trend may also reduce the value of a more robust design.

In particular for long-lived investments ($T^{\star\star} > \bar{T}_\mu, \bar{T}_\sigma$), lower uncertainty and faster climate change imply more robustness. For a stronger trend, it is beneficial to increase robustness because higher benefits can be obtained for a long time, which outweighs higher robustness costs. Higher uncertainty does not lead to increasing robustness for such investments, because the option value does not rise sufficiently (or even decreases) to justify additional robustness costs. These comparative statics of robustness with respect to uncertainty are thus in line with the option value decreasing once a certain threshold is crossed (cf. Proposition 7).

For comparatively short-lived investments ($T^{\star\star} < \bar{T}_\mu, \bar{T}_\sigma$), robustness would be decreased if there is lower uncertainty or faster climate change. For a stronger trend, higher robustness costs do not balance the marginal gains from robustness that are only achieved for a relative limited time. These short-lived investments, on the other hand, benefit from increasing robustness in light of higher uncertainty, since there are then more possibilities to gain from up-dated information during an extended lifetime.

With intermediate lifetimes between $\bar{T}_\mu$ and $\bar{T}_\sigma$, mixed cases are possible. Both thresholds give some indication about what time scales might make the difference between the cases (see Fig. 2). For the special case where the trend in the conditions μ roughly balances the discount rate r, the parameter β comes close to unity. Thus, if uncertainty is low, $\bar{T}_\sigma \approx (\beta\mu)^{-1} \approx 1/r$. For usual discount rates, the long-lived investments with the unconventional comparative statics are then those with economic lifetimes of more than 20 to 50 years. Even longer lifetimes are quite common, for instance, with transport infrastructure or buildings.

We can take parameters from other studies (with different objectives but parameterizing climate impacts as Brownian motion) to have an impression of possible thresholds. Guthrie (2021) studies implications of a process for inundation costs ($\mu = 0.048, \sigma = 0.03, r = 0.07$, so that $\bar{T}_\sigma \approx 32.6\,\text{yr}, \bar{T}_\mu \approx 14.5\,\text{yr}$), while Gersonius et al. (2013) investigate drainage infrastructure (the process for rainfall intensity uses $\mu = 1.8 \cdot 10^{-4}, \sigma^2 = 2.47 \cdot 10^{-4}, r = 0.035$, so that $\bar{T}_\sigma \approx 1071\,\text{yr}, \bar{T}_\mu \approx 4241\,\text{yr}$, clearly above life-times of up to 200 years).

We computed the isolines for both thresholds in Fig. 2. Suppose the relevant process is in a neighborhood of A, and the trends becomes stronger. Then, investments with a life-time of previously 30 years (above both thresholds) should become more robust. With more uncertainty, they should become less robust. The same would apply to an investment with 50 years life-time. Close to area C, the two investment's life times are below both thresholds, so they should be less robust for a stronger trend, and more robust for with rising uncertainty. We can also position mixed cases. Close to area B, the 30 years investment is above $\bar{T}_\mu$ but below $\bar{T}_\sigma$. Then, both a stronger trend and more uncertainty imply more robustness. For the 50 years investment, the comparative statics are like in A. In the neighborhood of D, the 50 years investment is above $\bar{T}_\mu$ and below $\bar{T}_\sigma$ (like the 30 years investment in B). Yet, now the 30 years investment behaves like in C.

We now assess changes of the expected lifetime if the investment's robustness is optimally chosen in the first stage (see Appendix for the proof). This requires that we add up the direct and indirect effects in Proposition 2 for our solution.

Proposition 9. *Assume that the conditions of Proposition 4 hold. Then, $\partial_c T^{\star\star} < 0$. If $T^{\star\star} \leq \bar{T}_\sigma$, then $\partial_{\sigma^2} T^{\star\star} > 0$. If $T^{\star\star} \leq \bar{T}_\mu$, then $\partial_\mu T^{\star\star} < 0$.*

Table 1
Summary of comparative statics results.

Optimal expected lifetime	Changing parameters	Effect on optimal	
		Robustness $a^\star$	Lifetime $T^{\star\star}$
Short			
$T^{\star\star} < \bar{T}_\mu$	trend μ	(−)	(−)
$T^{\star\star} < \bar{T}_\sigma$	uncertainty σ^2	(+)	(+)
Long			
$T^{\star\star} > \bar{T}_\mu$	trend μ	(+)	(?)
$T^{\star\star} > \bar{T}_\sigma$	uncertainty σ^2	(−)	(?)

For $T^{\star\star} > \bar{T}_\mu$, we do not obtain an analytical result for the effect of the trend, and similarly for $T^{\star\star} > \bar{T}_\sigma$ with respect to uncertainty. We summarize the main comparative statics results in Table 1. For comparatively long-lived investments, the effects of trends or uncertainty are ambiguous as expected in Section 2. If the chosen lifetime is relatively short, the direct and indirect effects on optimal lifetime (see Proposition 2), both due to trend and uncertainty, go in the same direction. Thus, the overall effect of a stronger trend is then negative while the one of higher uncertainty is positive. If the chosen lifetime is relatively long, the direct and indirect effects go in opposite directions. Unfortunately, we cannot prove the overall effect for the optimal expected lifetime analytically in this case.

When there is a stronger trend, there are gains from a lower cutoff value (a negative direct effect), as well as gains from decreasing robustness (a negative indirect effect), if the lifetime is relatively short. If the lifetime is relatively long, the gains from the now positive indirect effect might outweigh the gains from the direct effect, such that the expected optimal lifetime might rise. When there is higher uncertainty there are gains from a larger option value due to the positive direct and indirect effect, if the investment is relatively short-lived. If the lifetime is relatively long, a less robust design might outweigh a larger option value, such that the expected optimal lifetime is shortened. Our numerical experiments, however, lend to the hypothesis that the direct effect dominates: The optimal expected lifetime of particularly long-lived investments decreases further with a stronger trend, and increases further with higher uncertainty.

5. Numerical experiments

The analytical results show that the combined effects of the climate change trend and uncertainty on the expected lifetime of an optimally adapted investment can be ambiguous. For some cases, we can make clear analytical predictions, while in other cases the outcome depends on the solutions of implicit equations that do not allow for a closed-form representation. We thus explore these cases by means of numerical solutions.

Fig. 3 shows optimal robustness and optimal expected lifetime depending on specific other parameters. In accordance with Table 1, ranges with increasing and with decreasing robustness can be observed, separated by a trend μ where the optimal expected lifetime equals the threshold $\bar{T}_\mu$. Expected optimal lifetime decreases if $T^{\star\star} < \bar{T}_\mu$ as has been shown in Proposition 9. In the example, the optimal lifetime is generally decreasing for a stronger trend.

If investments with a comparatively long (expected optimal) lifetime are exposed to faster climate change, the additional costs of more robust design pay off. This is intuitive as the benefits from more robustness are obtained for a longer time. Yet, increasing robustness is not sufficient to completely compensate for faster climate change — the investment's lifetime becomes shorter and shorter. So, if the trend in the climatic conditions becomes even stronger, the lifetime becomes so short that more robustness is no longer justified. The decision rule switches to less robust designs in the presence of more severe climate change. Ultimately, a reduced lifetime is the necessary consequence.

Fig. 4 shows the optimal expected lifetime depending on both the trend and on uncertainty. Again, the different cases according to Table 1 can be observed. For lower uncertainty, robustness is increased such that also the optimal expected lifetime becomes longer. Although the investment becomes less robust for higher uncertainty, the lifetime is still extended. Thus, in the example, the optimal lifetime always rises with higher uncertainty.

If an investment is designed for relatively certain climatic conditions, the expected optimal lifetime is comparatively short. There are only limited reasons to keep an investment with negative current benefits running since the option value is low. If uncertainty is higher, a more robust design becomes beneficial to even out random fluctuations. In this case, the indirect effect of robustness on lifetime adds to the increasing option value, such that the lifetime is further extended. At some point, however, higher uncertainty leads to decreasing robustness. Although a longer (expected) lifetime might be a good reason to invest more robustly, two effects counterbalance this. First, there are diminishing returns from robustness in any case. Second, as shown in Proposition 7, the option value begins to decrease at some level of robustness. The overall effects on the lifetime remain positive. The (negative) indirect effect of reduced robustness is overcompensated for by the (positive) direct effect of uncertainty on the stopping time.

Fig. 4 also illustrates how the effect of uncertainty and the trend interact in a non-linear way. The second-stage decision links the first-stage robustness choice to the expected lifetime. The first stage can thus be conceived as choosing the optimal expected lifetime in light of the indirect effect's costs and benefits, adjusted by the option value.

Interestingly, we were not able to find parameter sets for a case in which faster climate change leads to longer optimal expected lifetimes, or one in which higher uncertainty implies shorter lifetimes. On the other hand, our experiments show the cases of both
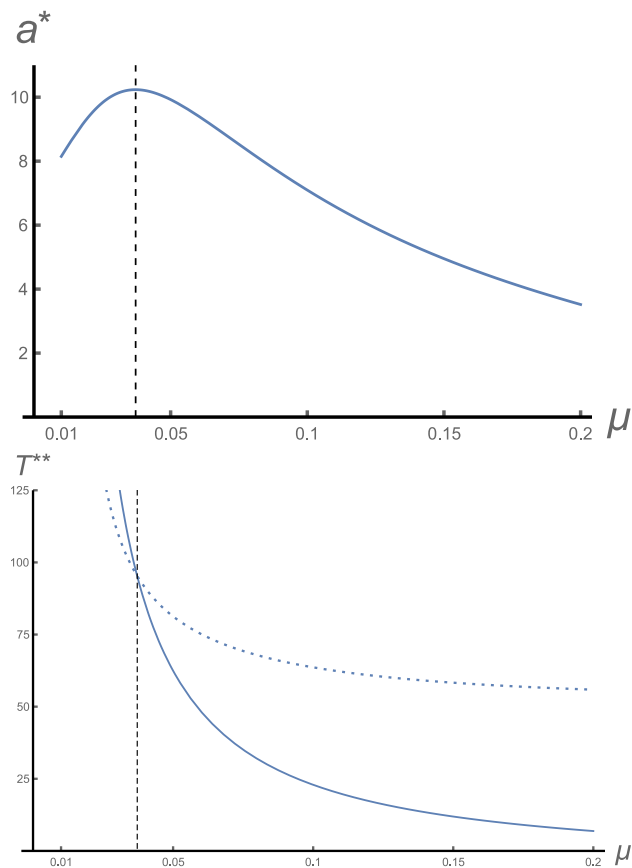
Fig. 3. Example for optimal robustness (upper panel) and optimal expected lifetime (lower panel) depending on the trend μ (with $r = 0.02, \sigma^2 = 0.02, c = \gamma = x_0 = 1.00$). The dotted curve represents $\bar{T}_\mu$ as a function of μ, at the dashed vertical line $T^{\star\star} = \bar{T}_\mu$.
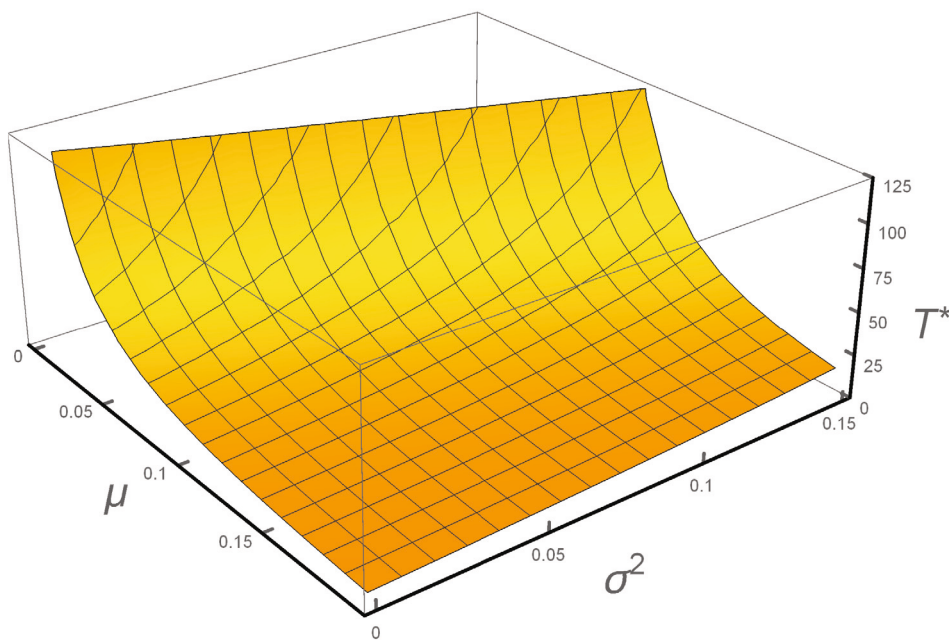

Fig. 4. Example for optimal expected lifetime depending on uncertainty σ^2 and trend μ (with $r = 0.02, c = \gamma = x_0 = 1.00$).

positive and negative effects of higher uncertainty and faster climate change on optimal robustness. Optimal lifetime yet always seems to increase with uncertainty, and decreases for stronger trends. The direct effects on lifetime seem to dominate the indirect effects.

6. Conclusion

This paper started from the question of whether the expected lifetime of an investment with an irreversible technical design should become shorter or whether the investment should be designed more robustly — if affected by detrimental climate change with a stronger trend or with higher uncertainty. We first analyzed the problem from a general perspective, and then focused on an application with geometric Brownian motion and a technical design parameter that can be interpreted as the investment's robustness.

The effects of the trend and of uncertainty on optimal investment design and lifetime can generally be decomposed into direct and indirect effects. For the direct effects, a stronger trend leads to ceteris paribus shorter lifetimes — the investment becomes unprofitable earlier. In line with real options theory, higher uncertainty ceteris paribus leads to stopping the investment at a later time due to an increasing premium to wait. The indirect effects are driven by the anticipated lifetime, which in turn depends on the strength of the trend, its uncertainty, and the chosen technical design. Conditions for the occurrence of the different cases can be identified. The case depends on whether the investment is comparatively short- or long-lived. In the short-lived cases, the direct effects dominate. One might say that if time does not matter much, everything is intuitive. In contrast, with comparatively long-lived investments, more complex and prima facie counterintuitive designs may be optimal. Higher uncertainty then leads to less robust designs, while stronger trends make more robust designs optimal. Our analysis thus shows some unexpected effects. The effects already appear for linear current benefits. They will feature in non-linear cases in an even more complex way. The effects we find have, to our knowledge, not been investigated in the theoretical literature so far.

The results and the approach taken are relevant for further applications not related to climate change adaptation. The debate on fossil stranded assets, for example, pictures a situation where investors face the decision of whether and how to continue investing in mining or conventional power plants in the presence of strong regulatory uncertainty (e.g. Pfeiffer et al., 2016; Kalkuhl et al., 2019; Eisenack et al., 2021; von Dulong et al., 2022). Retrofitting such power plants has limited value if policies, for instance, enforce a shut-down. This problem also applies to investment in renewables (some with shorter life-time than fossil fuel power plants) or to the expansion of electricity grids (long life-time). Also in the construction industry many design choices at the time of investment cannot be revised easily at a later point. This industry makes up a share of about 5% to 11% of GDP globally; and megaprojects might account for 8% of GDP (depending on the estimate, e.g., Crosthwaite, 2000; Flyvbjerg, 2014; World Economic Forum, 2016; OECD, 2016). The profitability of such investments depends on uncertain trends in demand, and might suffer from the risk of becoming technologically outdated. One might also consider utilities investments, new water or transport infrastructure components, in the light of uncertain demographic change or economic growth (cf. Chatterjee and Turnovsky, 2012). Their stream of benefits typically depends on the size of the population or the scale of economic activity that is served. Any investment in machinery must consider whether the design should be more specialized to efficiently meet a certain demand, or whether it should be more robust in the sense that its profits are less sensitive to the market conditions. Our model structure might also be motivated by considering investments with a shorter lifetime, such as a new computer: Does it make sense to choose a more expensive one that will have high performance for a longer period, or a cheaper one that will become outdated sooner?

While there seem to be many applications, there are also some limitations to our analysis which lend themselves naturally to extensions. Applying our general comparative statics results about the optimal lifetime for a concrete functional form, particularly for long-lived investments, turned out to be complicated. For other functional forms, the comparative statics results can likely not be evaluated analytically. Geometric Brownian motion was intentionally chosen as a case with substantial uncertainty. Further research could study other stochastic processes and other specifications of the current benefits. Our assumption of a completely irreversible design is admittedly quite polar in comparison to other irreversible investment studies which admit full flexibility in additional decision variables. Many applications are presumably between those extremes. We have chosen this assumption to bring the main effects clearly to light. In addition, it is conceptually not straightforward how to model intermediate kinds of flexibility. A more general model might include something like limited flexibility by considering subsequent investment cycles. A further interesting extension would be to consider risk aversion, as this would balance the effects from uncertainty and robustness in another way. Real-world applications and most such extensions would likely require simulation methods, as are common in the real options literature. Our study provides a template for doing so.

Although theoretical in nature, our results may be important for both private and public climate adaptation decisions about long-lived investments. Guidelines for wisely planning the design and lifetime of investments could help to avoid unnecessarily excessive expenditures for adaptation to the impacts of climate change. The general analysis presented in this paper offers the basis for applied numerical computations. A general take-home message for decision makers is that climate change uncertainty does not necessarily require more robust infrastructure, in particular if it is quite long-lived: Flexibility in terms of less robust investments can pay off.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Appendix. Mathematical appendix

Proof of Proposition 3 (*Optimal Stopping with Arbitrary Design*)**.** First show that the function Eq. (13) satisfies the Hamilton–Jacobi–Bellman (HJB) equation by differentiation. Note that

$$\mu x h' = \frac{\mu \gamma}{r} \frac{\beta}{\beta - 1} \left(\frac{x}{x^\star} \right)^\beta - \frac{\mu x}{a(r - \mu)}, \tag{A.1}$$

$$\frac{1}{2} \sigma^2 x^2 h'' = \frac{1}{2} \sigma^2 \frac{\gamma}{r} \beta \left(\frac{x}{x^\star} \right)^\beta, \tag{A.2}$$

so that the HJB equation evaluates to $\sigma^2 \beta^2 + (2\mu - \sigma^2)\beta - 2r$, which vanishes according to the definition of β. The positive root β can be explicitly written as $\beta = \frac{1}{\sigma^2} \left(((\mu - \frac{1}{2}\sigma^2)^2 + 2r\sigma^2)^{1/2} - (\mu - \frac{1}{2}\sigma^2) \right) > 0$. Also the value matching and smooth pasting conditions $h(x^\star) = h'(x^\star) = 0$ with $x^\star = \omega \gamma a$ are straightforward to verify. Now let $a > a_0$, which implies that $x^\star > x_0$, so that $h(x_0) > 0$. Together with Eq. (A.2) implying that $h'' > 0$, this shows that $x^\star$ is the global minimum of the value function.

If $a \le a_0$, we would obtain $h(x_0) \le 0$. Then the investment would be immediately be stopped (with $x^\star = x_0$). ■

Proof of Proposition 4 (*Optimal Design*)**.** As first main step, we show the **existence** of an inner maximum of $J(x_0, a) = h(x_0) - ca$ with respect to a. By using Eq. (15) and some re-arranging, the first-order condition can be written as

$$\partial_a J = -\frac{\gamma \beta}{r(\beta - 1)} a_0^\beta a^{-(1+\beta)} + \frac{x_0}{a^2(r - \mu)} - c = 0. \tag{A.3}$$

The second-order condition requires

$$\partial_{aa} J = \frac{\gamma \beta(\beta + 1)}{r(\beta - 1)} a_0^\beta a^{-(2+\beta)} - \frac{2x_0}{a^3(r - \mu)} < 0. \tag{A.4}$$

Yet, the second derivative $\partial_{aa} J$ vanishes at $a = \bar{a}$ (exactly once), since $\partial_{aa} J$ becomes negative if a exceeds $\bar{a}$. For $a < \bar{a}$, Eq. (A.4) cannot be satisfied.

We now show that the first and second-order condition will be satisfied at some point $a^\star > \bar{a}$ if the condition

$$c < \phi := \frac{\gamma^2 \omega^2}{(r - \mu)x_0} \left((\frac{2}{\beta + 1})^{\frac{2}{\beta - 1}} - (\frac{2}{\beta + 1})^{\frac{\beta + 1}{\beta - 1}} \right),$$

holds: Note that, by definition of a_0, the expected net value $J(x_0, a_0) = h(x_0) - ca_0 = h(x^\star) - ca_0 < 0$. Further, it can easily be seen from Eq. (A.3) that $\lim_{a \to \infty} J(x_0, a) \to -\infty$. Thus, since J is continuous, there must be a global inner maximum if J increases at least at one point $a > a_0$. This is indeed the case, since rearranging and using $c < \phi$ yields $\partial_a J(x_0, \bar{a}) > 0$.

This implies, beyond existence, that the inner maximum $a^\star > \bar{a}$, since $a^\star$ has to be to the right of the inflection point, where $J(a)$ is concave. Moreover, $a^\star > a_0$, since $\bar{a} = a_0 (\frac{2}{\beta + 1})^{\frac{1}{1 - \beta}} > a_0$ because $(\frac{2}{\beta + 1})^{\frac{1}{1 - \beta}} > 1$.

The assumption $c < \phi$ is also necessary. Suppose that $\partial_a J(x_0, \bar{a}) < 0$. Then J will remain decreasing above $\bar{a}$ since there is no further inflection point. Since J is also monotonically decreasing for $a < \bar{a}$, we would only obtain a corner solution $a = 0 < a_0$, i.e. the investment would be stopped immediately.

As second main step, we show that the optimum is **global and unique**. Let $a^\star > \bar{a} > a_0$ be an inner maximum of $J(a, x_0)$. We know that $x^\star > x_0$ and $J(x_0, a_0) < 0$ holds. Since $J(x_0, a^\star) > 0$ by assumption, $J(x_0, a^\star) > J(x_0, a_0)$, so that $a^\star$ is a global maximum. Furthermore, $a^\star$ is the only maximum, since $J(x_0, a)$ has only one inflection point with respect to a.

As last main step, we show that Eq. (17) **characterizes the optimal design**. Implicitly define z as the solution of

$$\left(\frac{2}{\beta + 1} \right)^{\frac{z(1+\beta)}{\beta - 1}} - \left(\frac{2}{\beta + 1} \right)^{\frac{2z}{\beta - 1}} = \frac{c}{\gamma} \frac{\mu - r}{\omega} a_0. \tag{A.5}$$

We first show that Eq. (17) yields the optimum, if some solution $z > 1$ exists. Afterwards, existence of z is shown. We know that a global maximum $a^\star > \bar{a}$ exists, and that it satisfies the first-order condition Eq. (A.3). Substitute $a^\star$ from Eq. (17) into Eq. (A.3) to obtain $\frac{\beta \gamma}{a_0 r(\beta - 1)} (\frac{2}{\beta + 1})^{z(-\frac{1+\beta}{1-\beta})} + \frac{x_0}{(r-\mu)a_0^2} (\frac{2}{\beta + 1})^{z(-\frac{2}{1-\beta})} - c = 0$. Rearranging gives Eq. (A.5). Moreover, since $z > 1$, we have $(\frac{2}{\beta + 1})^{\frac{z}{1 - \beta}} > (\frac{2}{\beta + 1})^{\frac{1}{1 - \beta}}$, so that $a^\star > \bar{a}$.

Now turn to existence and uniqueness of z. If there would be no $z > 1$ solving Eq. (A.5), then there would be no $a > \bar{a}$ solving Eq. (A.3), which would contradict the existence of the optimal $a^\star$. If there would be more than one $z > 1$ solving Eq. (A.5), then there would also be more than one $a > \bar{a}$ solving Eq. (A.3). This would then contradict the uniqueness of the optimal $a^\star$. ■

Proof of Proposition 5 (*Decision Without Uncertainty*). First show that the function Eq. (18) satisfies the HJB equation when $\sigma = 0$ by differentiation and substitution. Note that

$$\mu x h^{\circ\prime} = \frac{\gamma\mu}{r-\mu}(\frac{x}{x^\circ})^{\frac{r}{\mu}} - \frac{\mu}{a(r-\mu)}x, \tag{A.6}$$

$$-r h^\circ = -\frac{\gamma\mu}{r-\mu}(\frac{x}{x^\circ})^{\frac{r}{\mu}} + \frac{r}{a(r-\mu)}x - \gamma, \tag{A.7}$$

so that the HJB equation is satisfied. The value matching and smooth pasting conditions $h(x^\circ) = h'(x^\circ) = 0$ are straightforward to verify. Note also that

$$h^{\circ\prime\prime} = \frac{\gamma}{r-\mu}(\frac{r}{\mu}-1)(\frac{x}{x^\circ})^{\frac{r}{\mu}}x^{-2} > 0. \tag{A.8}$$

Furthermore $a > x_0/\gamma$ guarantees that $h^\circ(x_0) > 0$, so that x° is the global minimum of h°. If $a \le x_0/\gamma$, the investment would immediately be stopped with $x^\circ = x_0$. ∎

Proof of Proposition 6 (*Comparative Statics of Optimal Stopping with Arbitrary Design*). First turn to the derivatives of the cutoff value $x^\star = \omega\gamma a$, where $\omega > 1$ (according to Proposition 3). Thus, $\partial_a x^\star = \gamma\omega > 0$. It will be helpful to recall that $\partial_\mu\beta < 0$ (see standard literature, e.g. Dixit and Pindyck, 1994, which is extended here to $\mu > r$). Furthermore, $\partial_{\sigma^2}\beta > 0$ iff $\mu > r$, so that always $(r-\mu)\partial_{\sigma^2}\beta < 0$. We thus obtain

$$\partial_{\sigma^2}x^\star = -\gamma a\frac{r-\mu}{r(\beta-1)^2}\partial_{\sigma^2}\beta > 0,$$

and

$$\partial_\mu x^\star = \gamma a\frac{\alpha}{r(\beta-1)^2},$$

with $\alpha = -\partial_\mu\beta(r-\mu) - \beta(\beta-1)$. The sign of α is thus the same as the sign of $\partial_\mu x^\star$.

We show in the following that $\alpha < 0$. Substituting the explicit expressions for the root β and its partial derivative yields

$$\alpha = \frac{2\sigma^2\beta(-4r\sigma^2 - \sigma^4 - 4\mu^2 + (\sigma^2+2\mu)\sqrt{(\sigma^2-2\mu)^2 + 8r\sigma^2})}{4\sigma^4\sqrt{(\sigma^2-2\mu)^2 + 8r\sigma^2}}. \tag{A.9}$$

The denominator is always positive, and the outer bracket of the numerator can be rearranged to

$$(\sigma^2+2\mu)(\sqrt{(\sigma^2-2\mu)^2 + 8r\sigma^2} - (\sigma^2+2\mu)) + 4\sigma^2(\mu-r).$$

If can be verified by some equivalence transformations that this expression is negative iff $\frac{16\sigma^4(\mu-r)^2}{(\sigma^2+2\mu)^2} > 0$. The latter obviously holds. Thus, α and consequently $\partial_\mu x^\star$ are negative.

Finally, turn to the comparative statics for $E[T^\star]$. With respect to σ^2, μ, a, the signs of the derivatives can be determined from using the general results in Eq. (3), Eq. (4), Eq. (5), and recalling that $E[T^\star] = \frac{1}{\mu-\frac{1}{2}\sigma^2}\log\frac{x^\star}{x_0}$. The latter also yields $\partial_{x_0}E[T^\star] = -\frac{1}{(\mu-\frac{1}{2}\sigma^2)x_0 x^\star} < 0$. ∎

Proof of Proposition 7 (*Comparative Statics of Option Value*). The derivative

$$\partial_a\Theta(x) = \frac{\gamma}{a(r-\mu)}(\frac{x}{x^\circ})^{\frac{r}{\mu}}\left(1 - (\frac{(r-\mu)\beta}{r(\beta-1)})^{1-\beta}(\frac{x}{x^\circ})^{\beta-\frac{r}{\mu}}\right), \tag{A.10}$$

can be written as $\partial_a\Theta(x) = \alpha_1\alpha_2$ with $\alpha_1 := -\frac{\gamma}{r-\mu}(\frac{x}{x^\circ})^{\frac{r}{\mu}}\frac{1}{a}$, and $\alpha_2 := \omega^{1-\beta}(\frac{x}{x^\circ})^{\beta-\frac{r}{\mu}} - 1$. If $r < \mu$, then $\alpha_1 > 0$. Due to Eq. (16), $\beta - \frac{r}{\mu} > 0$, so that $\alpha_2 > 0$ is equivalent to the expression $\omega^{\frac{\mu(1-\beta)}{r-\mu\beta}} < \frac{x}{x^\circ}$. If $r > \mu$, then $\alpha_1 < 0$. Due to Eq. (16), $\beta - \frac{r}{\mu} < 0$, so that $\alpha_2 < 0$ is also equivalent to the same expression. Finally, since $x^\circ = \gamma a$, the inequality yields Eq. (22). ∎

Proof of Proposition 8 (*Comparative Statics of Robustness with Optimal Design*). We know from Eq. (8) and Eq. (9) that $\partial_{a\mu}a^\star \doteq \partial_{a\mu}J(x_0, a^\star, \mu, \sigma^2)$ and $\partial_{\sigma^2}a^\star \doteq \partial_{a\sigma^2}J(x_0, a^\star, \mu, \sigma^2)$. In addition,

$$\partial_c a^\star \doteq \partial_{ac}J(x_0, a^\star, \mu, \sigma^2, c) = -1 < 0.$$

Considering the first-order condition for optimal design and the definition of a_0 in Proposition 3, the following holds:

$$\partial_{a\sigma^2}J = \partial_{a\sigma^2}h(x_0) = -\frac{\gamma}{r(\beta-1)}a^{-\beta-1}a_0^\beta\partial_{\sigma^2}\beta(1-\beta\ln(\frac{a}{a_0})),$$

$$\partial_{a\mu}J = \frac{\gamma}{r(\beta-1)}a^{-\beta-1}a_0^\beta(\beta\partial_\mu\beta\ln(\frac{a}{a_0}) - \frac{\beta^2}{r-\mu} - \partial_\mu\beta) + \frac{x_0}{a^2(r-\mu)^2}$$

$$= \frac{\gamma\omega}{(r-\mu)^2}\frac{1}{a}\left(\frac{a_0}{a} - (r-\mu)(\frac{a_0}{a})^\beta\partial_\mu\beta\ln(\frac{a_0}{a}) - \beta(\frac{a_0}{a})^\beta - \frac{r-\mu}{\beta}(\frac{a_0}{a})^\beta\partial_\mu\beta\right)$$

$$\doteq (\frac{a}{a_0})^{\beta-1} + (r-\mu)\partial_\mu\beta\ln(\frac{a}{a_0}) - \beta - \frac{r-\mu}{\beta}\partial_\mu\beta$$

Recall from the proof of Proposition 6 that $(r - \mu)\partial_{\sigma^2}\beta < 0$, so that Eq. (16) implies $\frac{\partial_{\sigma^2}\beta}{\beta - 1} < 0$. Thus, $\partial_{a\sigma^2}J \doteq 1 - \beta \ln(\frac{a}{a_0})$.

In order to further determine the signs of the derivatives, we introduce the following change of variables: By recalling Eq. (15) and $T^{\star\star} = E[T^\star] = \frac{1}{\mu - \frac{1}{2}\sigma^2}\log\frac{x^\star}{x_0} = \frac{1}{\mu - \frac{1}{2}\sigma^2}\log\frac{a^\star}{a_0}$, we obtain $\ln(\frac{a^\star}{a_0}) = (\mu - \frac{1}{2}\sigma^2)T^{\star\star}$ and $\frac{a^\star}{a_0} = \exp(\mu - \frac{1}{2}\sigma^2)T^{\star\star}$. Then, if evaluated at $a = a^\star$, we obtain $\partial_{a\sigma^2}J \doteq 1 - \beta(\mu - \frac{1}{2}\sigma^2)T^{\star\star}$. The right-hand side is decreasing in $T^{\star\star}$ with the zero at $T^{\star\star} = \bar{T}_\sigma$. Thus, $\partial_{a\sigma^2}J$ is positive at $a = a^\star$ iff $T^{\star\star} < \bar{T}_\sigma$.

Now, apply the change of variables again and re-arrange to obtain

$$\partial_{a\mu}J(a^\star) \doteq (r - \mu)\partial_\mu\beta\left((\mu - \frac{1}{2}\sigma^2)T^{\star\star} - \frac{1}{\beta}\right) + e^{(\mu - \frac{1}{2}\sigma^2)(\beta - 1)T^{\star\star}} - \beta. \tag{A.11}$$

We now show that this expression has exactly one positive root for $T^{\star\star}$, denoted by $\bar{T}_\mu$. If it is shown that $\bar{T}_\mu > 0$ is well-defined, and that Eq. (A.11) is negative iff $T^{\star\star} < \bar{T}_\mu$, we can conclude that $\partial_{a\mu}J(a^\star) > 0$ iff $T^{\star\star} > \bar{T}_\mu$, and the proof is finished.

For this last step, use that Eq. (A.11) is continuous in $T^{\star\star}$. Observe that the second derivative of Eq. (A.11) with respect to $T^{\star\star}$ is

$$(\mu - \frac{1}{2}\sigma^2)^2(\beta - 1)^2 e^{(\mu - \frac{1}{2}\sigma^2)(\beta - 1)\tau} > 0,$$

i.e. Eq. (A.11) is strictly convex in $T^{\star\star}$. Furthermore, evaluate Eq. (A.11) at $T^{\star\star} = 0$ to obtain

$$-\frac{r - \mu}{\beta}\partial_\mu\beta + 1 - \beta.$$

By substituting the explicit expressions for $\partial_\mu\beta, \beta$, this is equivalent to

$$\frac{(\sigma^2 + 2\mu)(\sqrt{(\sigma^2 - 2\mu)^2 + 8r\sigma^2} - (\sigma^2 + 2\mu)) + 4\sigma^2(\mu - r)}{2\sigma^2\sqrt{(\sigma^2 - 2\mu)^2 + 8r\sigma^2}},$$

which is, in turn, equivalent to α from Eq. (A.9). It has been shown in the proof of Proposition 6 that $\alpha < 0$, so that Eq. (A.11) is always negative at $T^{\star\star} = 0$. We can thus summarize that Eq. (A.11) is convexly increasing from some negative value, so it needs to vanish exactly once, and becomes positive for higher values of $T^{\star\star}$. ∎

Proof of Proposition 9 (*Comparative Statics of Expected Life-time with Optimal Design*). Consider Eq. (10) - Eq. (11) and the comparative statics results from Proposition 6 and Proposition 8. If $T^{\star\star} \leq \bar{T}_\sigma$, then

$$\partial_{\sigma^2}T^{\star\star}(x_0, \mu, \sigma^2) = \partial_{\sigma^2}E[T^\star] + \partial_a E[T^\star] \cdot \partial_{\sigma^2}a^\star > 0, \tag{A.12}$$

and if $T^{\star\star} \leq \bar{T}_\mu$, then

$$\partial_\mu T^{\star\star}(x_0, \mu, \sigma^2) = \partial_\mu E[T^\star] + \partial_a E[T^\star] \cdot \partial_\mu a^\star < 0. \tag{A.13}$$

Finally, $\partial_c T^{\star\star}(x_0, \mu, \sigma^2, c) = \partial_a E[T^\star] \cdot \partial_c a^\star < 0$. ∎

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